FOR THE
IB DIPLOMA
PROGRAMME

Mathematics

APPLICATIONS AND INTERPRETATION **HL**

EXAM PRACTICE WORKBOOK

Paul Fannon
Vesna Kadelburg
Stephen Ward

Boost
HODDER EDUCATION

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ISBN: 9781398321885

First published in 2021 by

Hodder Education,
An Hachette UK Company
Carmelite House
50 Victoria Embankment
London EC4Y 0DZ

www.hoddereducation.com

Impression number 10 9 8 7 6 5 4 3 2 1

Year 2023 2022 2021

Illustrations by Ben Woolley and Aptara Inc.

Typeset in India by Aptara, Inc.

Printed in the UK

A catalogue record for this title is available from the British Library.

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Introduction

Revising for exams can sometimes be overwhelming. This book is designed to help you structure your revision and provide opportunities to practise sitting exam-style papers. Revision should be a cycle of going through your notes and textbooks, practising exam-style questions, reviewing your strengths and weaknesses, and returning to your notes and textbooks.

There are five skills that are needed for exam success:

- knowledge of all the topics required in the syllabus
- facility with basic algebra and arithmetic
- familiarity with your calculator
- the ability to make links and solve problems
- calmness under test conditions.

You need to be aware of your own strengths and weaknesses in each of these areas. This revision guide is designed to help with each of them.

How to use this book

The book comprises four sections that are designed to help you master the five skills listed above.

Calculator checklist

This lists all the tools provided by your GDC (graphical display calculator) that you need to be familiar with. Different calculators might have slightly different input methods, so it is best to use your own calculator manual (these can be found online) to find out the exact syntax yours uses.

Syllabus revision

This section goes through the syllabus line by line to make sure you have covered every part thoroughly. Each skill required in the syllabus is exemplified by a question. You can either start by going over the syllabus content, or by doing the questions. These questions illustrate the basic skills you need; however, they are not all designed to be exam-style questions as they are designed to check syllabus understanding rather than problem-solving. The answers to these questions can be found online at www.hoddereducation.co.uk/ib-extras. Once you are happy, tick the 'revised' box. If you need more details, there are references to the section in the accompanying Hodder Education *Mathematics for the IB Diploma: applications and interpretation SL* or *Mathematics for the IB Diploma: applications and interpretation HL* textbook corresponding to each syllabus item.

Paper plan

This table provides an overview of the entire syllabus that maps the practice papers in this book and in the *Mathematics for the IB Diploma: applications and interpretation HL* textbook to the different topics and also serves as a revision checklist. You should use the mastery section to tick off and make sure that you have covered each topic. When you have revised the topic, you can tick the second column. Then try doing some questions – either from the textbook or the practice papers – and tick the final column once you feel confident with the topic.

The practice paper section shows the corresponding topic for each question in the textbook practice papers and the sets of practice papers in this book. You can use this to check the type of questions that you might get on each topic.

Practice papers and mark schemes

The best way to practise for exams is to do exams. These papers are designed to mimic the style of real IB papers. The questions often combine more than one syllabus topic and can require you to make novel links. As in the real exam papers, there is space for you to write in your calculations and answers to questions in Paper 1; for Paper 2, you will need to use a separate notebook.

Understanding mark schemes

Once you have tried some of the practice papers in this book, it is a very good idea to mark your own (and also mark other people's) to see what makes things easy or difficult to award marks.

There are three different types of marks awarded in exams:

M These are method marks. They are awarded for a clear and obvious attempt to use the correct method. There is a certain amount of subjective opinion needed to award these. For example, if you are asked to find the length of the hypotenuse, h, of a right-angled triangle with shorter sides 5 and 7, which of the following would be awarded a method mark?

I $h = 5 + 7 = 12$

II $h = \sin(5) + \cos(7) = 1.08$

III $h = \sqrt{5 + 7} = 3.46$

IV $h = 5^2 + 7^2 = 74$

V $h = \sqrt{7^2 - 5^2} = 4.90$

VI $h = \sqrt{5^2 + 7^2} = 5 + 7 = 12$

Most examiners would agree that the first three examples are not good enough to award a method mark. In case VI, even though there is subsequent incorrect working and the wrong answer, a method mark would still be awarded. Cases IV and V are on the boundary of what might be acceptable and would probably require discussion among the examiners to find a clear boundary, but it is likely both answers would be awarded a method mark. However, an answer of 74 or 4.90 by itself would not be awarded any marks because, even though we might have suspicions about where these numbers have come from, it has not been clearly communicated.

Sometimes method marks have brackets around them, for example, **(M1)**. In this case they do not have to be explicitly seen and can be inferred from the correct answer.

Remember that sometimes the question requires a particular method (for example, find the maximum value of the function by differentiating) or it might require you to explicitly use the previous working (generally indicated by using the word 'hence'). If you use a different method in either of these instances, even if it works, you will not normally gain any credit.

Many questions will be answered primarily by using a calculator. However, you can still get some method marks for communicating what you are doing. Remember to write down any numbers that you put into your calculator that are not given in the question (for example, midpoints of grouped data). If you are using a graph to solve an equation, then draw a quick sketch.

A These are accuracy marks. They are for obtaining the correct answer. If there is a previous method mark without a bracket around it then these marks can only be awarded if the previous method mark was awarded (this tends to happen in situations where examiners think the correct answer can be guessed so they need to see supporting evidence, or when the question was a 'show that' or 'proof' question, where the communication rather than just the final answer is assessed).

Often lines are denoted M1A1 – this means one method mark for a correct attempt and one accuracy mark for doing it correctly.

The accuracy mark is withheld if the value or expression is wrong; however, it can also be withheld if the answer is written in calculator notation (for example, 1.8*E*9 rather than 1.8×10^9) or is given to the wrong accuracy – remember that all final answers should be given either exactly or written to three significant figures unless the question says otherwise. It is usually a good idea to write down intermediate working to more significant figures to ensure that the final answer is correct to at least three significant figures (and ideally store the answer to the full accuracy your calculator can hold using the calculator memory store).

Accuracy marks are also awarded when sketching graphs. It is important to choose an appropriate window to show all the important features of the graph and to label any relevant features (for example, axis intercepts, turning points, asymptotes).

Unless a particular form is required, most equivalent forms are accepted – for example, $x^2 + x$ or $x(x + 1)$ would normally both be fine. However, there is also an expectation that you understand the general requirements of the course. For example, if the question asked you to find the area under the curve $y = x^2$ between 0 and 1, the answer $\int_0^1 x^2 dx$, while *technically* equivalent to the correct answer, is not sufficiently simplified – the acceptable answer would be $\frac{1}{3}$ or 0.333.

R These are marks awarded for clear reasoning – often in response to a question asking for explanation or justification. They might also be used when choosing particular solutions from equations (for example, saying that the solution of a quadratic that is negative cannot be a probability).

You may also see an **AG** notation in the mark schemes. This is when the answer is given in the question and it is just a reminder to the examiner that the correct answer does not mean that people have reasoned properly and to be particularly watchful for flawed arguments that just happen upon the right answer.

Sometimes later parts of the question use an answer from a previous part. If you got the earlier part of the question wrong, the examiner will try to award 'follow through' marks by checking whether your method works for your prior incorrect answer. However, follow through marks may only be awarded if you have clearly communicated how you are using your previous answer, if you have not fundamentally changed the nature of the question (for example, solving a quadratic equation turned into solving a linear equation) and if your answer is not implausible (for example, a negative probability).

Revision tips

- Do not leave all your revision until the last minute. The IB is a two-year course with many later topics building on previous topics. One psychological study suggested that you need to 'learn' something seven times for it to be really fixed in your mind. Try to use each class test, mock exam or new topic as an opportunity to revise previous work.
- Revision should be an active rather than a passive process – often you can read through notes for hours and gain very little new knowledge. Try to do some questions first, then read through your notes and textbooks once you get stuck. Your reading will be far more focused if you are trying to find the solution to a particular difficulty.
- Try varied memorization strategies until you find one that works for you – copying out pages of notes does not work for most people. Strategies that do work for some people include using colour to prioritize key facts, using mind maps and making up silly songs to memorize techniques. Psychologists have found a strong link between memory and smell, so you could try using a particular perfume or deodorant while revising, then using the same one in the final exam!
- Work positively with others – some group revision can be a great way of improving your understanding as you can bounce ideas off each other, try to explain a concept to someone who is struggling or design exam-style questions for your friends to do. However, do be careful – avoid feeling bad by comparing yourself to people who seem to be good at everything and do not be tempted to feel good about yourself by making others feel bad – neither scenario is productive.
- Practise checking your answers. This is a vital skill that you will not suddenly be able to do in the final exam if you never do it in your revision. Think about good ways to check answers; for example, with and without a calculator, working backwards and sanity checking that the answer is plausible.
- Become an expert at using your exam calculator. You cannot start working on this skill too early, as each calculator has its own quirks that you need to get used to. Make sure you are using it in the mode required for the exam and know what happens when the memory is cleared and it is reset ahead of the exam; for example, does it default to radians or degrees mode?
- Become familiar with the exam formula booklet. It has lots of useful information, but only if you are used to using it – make sure you know what all the symbols mean and where everything is, well ahead of your final exam. Formulae that are included in the formula booklet are indicated in the syllabus content sections of this book by this icon:
- Make sure some of your revision is under timed conditions. During the exam, the time flashes by for some people whereas others have to pace themselves or they run out of steam towards the end of an exam.
- Do not get downhearted if you are getting lots of things wrong, especially at the beginning of the revision process. This is absolutely normal – in fact, you learn a lot more from the things which you get wrong than the things you get right!

- Weirdly, too much revision can actually be counterproductive. You will have your own personal concentration span beyond which there is no point revising without a small break. Check that your revision plan is achievable, and schedule in plenty of relaxation time.
- Try to get into stable sleeping and eating patterns in the run-up to the exam. If you are getting up each day at noon and never having caffeine, then a 9am exam with lots of coffee is unlikely to go well!
- Unless you know that you only have a very short-term memory, it is unlikely that the night before an exam is the best time to revise. Going for a run, doing some yoga or reading a good book and having a good night's sleep is likely to be worth far more marks than last minute panic revision.
- If you choose to do any revision between Paper 1 and Paper 2, do use the syllabus checklist to see if there are any major topics not covered in the first paper and focus your revision on those.

Exam tips

- Use the reading time wisely. Every mathematics exam starts with five minutes of reading time in which you are not allowed to write. This time is vital – make sure you read as much of the paper as you can and mentally start making a plan.
- The examiners design the difficulty of the questions to be in increasing order in the short questions, and in increasing order within and between each long question; however, their judgement of difficulty is not always going to align with yours, so do not assume that you should do the questions in order. Many people try all the short questions first, spend too long on the last, often tricky, short question and then either panic or run out of time on the long questions. Sometimes the first long question is the easiest question on the paper, so consider doing that first. There is no substitute for potentially gaining lots of marks early on to build confidence for the rest of the exam.
- Keep an eye on the time. Each mark equates to approximately one minute – so do not spend 10 minutes on a question worth only 2 marks. Sometimes you have to abandon one question and move on to the next.
- Do not get dispirited if you cannot do a question – the exam is meant to be challenging and you will not need 100% of the marks to get the grade you are aiming for. The worst thing you can do is let one bad question put you off showing your ability in other questions.
- Look at the mark schemes to understand what is given credit. Even when many method marks are implied, only putting down the final answer is a high-risk strategy! Even the best mathematicians can make mistakes entering numbers into calculators. Mathematical communication is an important skill so try to convey your reasoning clearly – this has the advantage of enabling you to score some marks even if you make a mistake and of marshalling your ideas so you are more likely to get the answer right in the first place.
- Especially in the long questions, do not assume that just because you cannot do an early part you cannot do later parts. Even if you get an early part wrong, follow through marks may still be available in later parts if you clearly communicate the correct method, even if you are using the wrong numbers. Sometimes the later parts of questions do not need the results from earlier parts anyway. The only way that you can guarantee getting no marks for part of a question is by leaving it blank!
- In Paper 2, identify which questions are 'calculator questions'. Too many people try to do these questions using non-calculator techniques that do work, but often absorb a lot of time.
- Keeping the exam in perspective is perhaps more important than anything else. While it is of some importance, always remember that exams are artificial and imperfect measurements of ability. How much you can achieve working in silence, under timed conditions and by yourself without any resources on one particular set of questions is not what is most valued in mathematics. It should not be the only outcome from the course that matters, nor should it be how you judge yourself as a mathematician. It is only when you realize this that you will relax and have a chance of showing your true ability.
- Finally, make sure that you understand the command terms used in exams – these are listed below. In particular, 'write down' means you should be able to answer without any major work – if you find yourself doing lots of writing then you have missed something!

Command term	Definition
Calculate	Obtain a numerical answer showing the relevant stages in the working.
Comment	Give a judgment based on a given statement or result of a calculation.
Compare	Give an account of the similarities between two (or more) items or situations, referring to both (all) of them throughout.
Compare and contrast	Give an account of similarities and differences between two (or more) items or situations, referring to both (all) of them throughout.
Construct	Display information in a diagrammatic or logical form.
Contrast	Give an account of the differences between two (or more) items or situations, referring to both (all) of them throughout.
Deduce	Reach a conclusion from the information given.
Demonstrate	Make clear by reasoning or evidence, illustrating with examples or practical application.
Describe	Give a detailed account.
Determine	Obtain the only possible answer.
Differentiate	Obtain the derivative of a function.
Distinguish	Make clear the differences between two or more concepts or items.
Draw	Represent by means of a labelled, accurate diagram or graph, using a pencil. A ruler (straight edge) should be used for straight lines. Diagrams should be drawn to scale. Graphs should have points correctly plotted (if appropriate) and joined in a straight line or smooth curve.
Estimate	Obtain an approximate value.
Explain	Give a detailed account including reasons or causes.
Find	Obtain an answer showing relevant stages in the working.
Hence	Use the preceding work to obtain the required result.
Hence or otherwise	It is suggested that the preceding work is used, but other methods could also receive credit.
Identify	Provide an answer from a number of possibilities.
Integrate	Obtain the integral of a function.
Interpret	Use knowledge and understanding to recognize trends and draw conclusions from given information.
Investigate	Observe, study, or make a detailed and systematic examination, in order to establish facts and reach new conclusions.
Justify	Give valid reasons or evidence to support an answer or conclusion.
Label	Add labels to a diagram.
List	Give a sequence of brief answers with no explanation.
Plot	Mark the position of points on a diagram.
Predict	Give an expected result.
Prove	Use a sequence of logical steps to obtain the required result in a formal way.
Show	Give the steps in a calculation or derivation.
Show that	Obtain the required result (possibly using information given) without the formality of proof. 'Show that' questions do not generally require the use of a calculator.
Sketch	Represent by means of a diagram or graph (labelled as appropriate). The sketch should give a general idea of the required shape or relationship, and should include relevant features.
Solve	Obtain the answer(s) using algebraic and/or numerical and/or graphical methods.
State	Give a specific name, value or other brief answer without explanation or calculation.
Suggest	Propose a solution, hypothesis or other possible answer.
Verify	Provide evidence that validates the result.
Write down	Obtain the answer(s), usually by extracting information. Little or no calculation is required. Working does not need to be shown.

Some advice about Paper 3

Paper 3 is a new type of examination to the IB. It will include long, problem-solving questions. Do not be intimidated by these questions – they look unfamiliar, but they will be structured to guide you through the process. The early parts will often suggest tools in simple situations which you then have to apply to more complicated situations.

All of the parts have their own challenges, so it is not always the case that questions get harder as you go on (although there might be a trend in that direction). Do not assume that just because you cannot do one part you should give up – there might be later parts that you can still do.

Some parts might look very unfamiliar, and it is easy to panic and think that you have just not been taught about something. However, one of the skills being tested is mathematical comprehension so it is possible that a new idea is being introduced. Stay calm, read the information carefully and be confident that you do have the tools required to answer the question, it might just be hidden in a new context.

These questions are meant to be interlinked, so if you are stuck on one part try to look back for inspiration. This might be looking at the answers you have found, or it might be trying to reuse a method suggested in an earlier part. Similarly, even more than in other examinations, it is vital in Paper 3 to read the whole question before you start to answer. Sometimes later parts will clarify how far you need to go in earlier parts, or give you ideas about what types of method will be useful in the question.

These papers will almost certainly have parts which focus on interpretation of the mathematical results. Try to practise extracting meaning from mathematical results and describing it succinctly.

These questions are meant to model the thinking process of mathematicians. Perhaps the best way to get better at them is to imitate the mathematical process at every opportunity. So the next time you do a question, see if you can spot hidden assumptions, critique the validity of the process and find inferences from your results. The more you do this, the better you will become.

Calculator checklist

You should know how to:

	Skill	Got it!	Need to check
General	Change between radian and degrees mode.		
	Set output to 3 significant figures.		
	Store answers in calculator memory.		
	Edit previous calculations.		
Number and algebra	Input and interpret outputs in standard index form ($a \times 10^n$).		
	Use the sequence functions to find terms of an arithmetic and geometric sequence.		
	Use tables to display sequences.		
	Use the sum function to sum sequences.		
	Use the TVM package to answer questions about compound interest and depreciation, including finding unknown interest rates and interest periods.		
	Use the TVM package to answer questions about amortization and annuities.		
	Evaluate logarithms to base 10 and e.		
	Solve polynomial equations.		
	Solve simultaneous equations.		
	Input complex numbers in both Cartesian and modulus-argument forms.		
	Change between the two forms of complex numbers.		
	Perform calculations with complex numbers.		
	Input a matrix.		
	Perform calculations with matrices, including powers, determinant and inverse.		
Functions	Graph equations of the form $y = \mathrm{f}(x)$.		
	Use the zoom / window functions to explore interesting features of graphs.		
	Use the trace function to explore graphs, especially suggesting asymptotes.		
	Find axis intercepts of graphs.		
	Find the coordinates of local maxima or minima of graphs.		
	Find the points of intersection of two graphs.		
	Solve equations using solve functions on the calculator.		
Statistics and probability	Input data, including from frequency tables and grouped data.		
	Find mean, median, mode, quartiles and standard deviation from data.		
	Select the correct standard deviation.		
	Input bivariate data.		
	Find Pearson correlation coefficient of data.		
	Find the equation of a regression line.		
	Conduct non-linear regression for quadratic, cubic, exponential, power and sine models, included the calculation of the R^2 value.		
	Calculate probabilities for a given binomial distribution.		
	Calculate probabilities for a given Poisson distribution.		
	Calculate probabilities for a given normal distribution.		
	Calculate boundary values from probabilities for a given normal distribution.		
	Find the confidence interval for the mean (using normal or t-distribution).		
	Conduct chi-squared goodness of fit tests.		
	Conduct chi-squared contingency table tests.		
	Conduct t-tests on a single sample or two samples.		
	Conduct a hypothesis test for the mean.		
	Conduct a hypothesis test for the correlation coefficient.		
Calculus	Estimate the value of a limit from a table or a graph.		
	Find the derivative of a function at a given point.		
	Use a calculator to sketch the derivative of a function.		
	Find definite integrals.		
	Find areas using definite integrals.		
	Generate a phase portrait for a system of differential equations.		

Syllabus revision

1 Number and algebra

Syllabus content

S1.1	Standard form	
	Book Section 1B	Revised ☐
Syllabus wording	*You need to be able to:*	*Question*
Operations with numbers of the form $a \times 10^k$ where $1 \le a < 10$.	Input and interpret numbers of this form on the calculator.	1 ☐
	Factorize to add or subtract numbers in standard form.	2 ☐
	Use the laws of exponents when multiplying or dividing numbers in standard form.	3 ☐

S1.2	Arithmetic sequences and series	
	Book Section 2A	Revised ☐
Syllabus wording	*You need to be able to:*	*Question*
Use of the formulae for the *n*th term and the sum of the first *n* terms of the sequence.	Find the *n*th term of an arithmetic sequence. Use: $u_n = u_1 + (n-1)d$	4 ☐
	Use the formula to determine the number of terms in an arithmetic sequence.	5 ☐
	Set up simultaneous equations to find the first term and common difference.	6 ☐
	Find the sum of *n* terms of an arithmetic sequence. There are two formulae in the formula booklet. You should be able to use: $S_n = \frac{n}{2}(2u_1 + (n-1)d)$	7 ☐
	Or use: $S_n = \frac{n}{2}(u_1 + u_n)$	8 ☐
Use of sigma notation for sums of arithmetic sequences.	Understand how sigma notation relates to arithmetic sequences.	9 ☐
	Evaluate expressions using sigma notation.	10 ☐
Applications.	Recognize arithmetic sequences from descriptions.	11 ☐
	In particular, be aware that simple interest is a type of arithmetic sequence.	12 ☐
Analysis, interpretation and prediction where a model is not perfectly arithmetic in real life.	Find the common difference as an average of the differences between terms.	13 ☐

S1.3	Geometric sequences and series	
	Book Section 2B	Revised ☐
Syllabus wording	*You need to be able to:*	*Question*
Use of the formulae for the *n*th term and the sum of the first *n* terms of the sequence.	Find the *n*th term of a geometric sequence. Use: $u_n = u_1 r^{n-1}$	14 ☐
	Use the formula to determine the number of terms in a geometric sequence.	15 ☐
	Set up simultaneous equations to find the first term and common ratio.	16 ☐
	Find the sum of *n* terms of a geometric sequence using: $S_n = \frac{u_1(r^n - 1)}{r-1} = \frac{u_1(1-r^n)}{r-1},\ r \neq 1$	17 ☐
Use of sigma notation for sums of geometric sequences.	Understand how sigma notation relates to geometric sequences.	18 ☐
	Evaluate expressions using sigma notation.	19 ☐
Applications.	Recognize geometric sequences from descriptions.	20 ☐

S1.4 Financial applications of geometric sequences

Book Section 2C | Revised ☐

Syllabus wording	You need to be able to:	Question
Financial applications of geometric sequences and series. • Compound interest. • Annual depreciation.	Calculate values of investments with compound interest using financial packages or using: $FV = PV \times \left(1 + \frac{r}{100k}\right)^{kn}$ where: FV is the future value PV is the present value n is the number of years k is the number of compounding periods per year r% is the nominal annual rate of interest.	21 ☐
	Calculate interest rates required for particular outcomes.	22 ☐
	Calculate the number of periods required for a particular outcome.	23 ☐
	Calculate the value of goods suffering from depreciation.	24 ☐
	Calculate the real value of investments after inflation.	25 ☐

S1.5 Exponents and logarithms

Book Section 1A, 1C | Revised ☐

Syllabus wording	You need to be able to:	Question
Laws of exponents with integer exponents.	Evaluate expressions involving integer exponents including using: $a^m \times a^n = a^{m+n}$ $\frac{a^m}{a^n} = a^{m-n}$ $(a^m)^n = a^{mn}$ $a^{-n} = \frac{1}{a^n}$ $(ab)^n = a^n \times b^n$ $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$	26 ☐
	Simplify algebraic expressions using the above rules.	27 ☐
Introduction to logarithms with base 10 and e.	Use the fact that $a^x = b$ is equivalent to $\log_a b = x$	28 ☐
	Know that natural logarithms, in x, are equivalent to $\log_e x$ where e = 2.718…	29 ☐
Numerical evaluation of logarithms using technology.	Use your calculator to evaluate logarithms to base 10 and e.	30 ☐

S1.6 Approximation and estimation

Book Section 11A | Revised ☐

Syllabus wording	You need to be able to:	Question
Approximation: decimal places, significant figures.	Round to a given number of decimal places.	31 ☐
	Round to a given number of significant figures.	32 ☐
	Choose an appropriate degree of accuracy for an answer.	33 ☐
Upper and lower bounds.	Report upper and lower bounds of rounded numbers as an inequality.	34 ☐
Percentage errors.	Calculate percentage errors using: $\epsilon = \left\|\frac{v_A - v_E}{v_E}\right\| \times 100\%$ where v_E is the exact value and v_A is the approximate value.	35 ☐
	Find the maximum percentage error caused by rounding.	36 ☐
Estimation.	Determine if an answer is reasonable.	37 ☐

S1.7	Amortization and annuities using technology		
	Book Section 11B	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*	
Amortization and annuities using technology.	Calculate the outstanding amount of loans being regularly paid off.	38	☐
	Calculate the value of investments with regular contributions made.	39	☐
	Calculate the amount of annuity that can be purchased.	40	☐

S1.8	Solving equations		
	Book Section 12A, 12B	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*	
Use technology to solve systems of linear equations in up to three variables.	Recognize linear simultaneous equations and solve them using the appropriate GDC function.	41	☐
Use technology to solve polynomial equations.	Recognize the order of a polynomial equation, understand the terms zero and root and solve polynomial equations.	42	☐

H1.9	Laws of logarithms		
	Book Section 1B	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*	
Laws of logarithms.	Manipulate logarithms algebraically using: $\log_a xy = \log_a x + \log_a y$; $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$; $\log_a x^m = m\log_a x$ (in exams a will be either 10 or e).	43	☐

H1.10	Rational exponents		
	Book Section 1A	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*	
Simplifying expressions, both numerically and algebraically, involving rational exponents.	Simplify numerical expressions involving rational exponents.	44	☐
	Simplify algebraic expressions involving rational exponents.	45	☐

H1.11	Sum of an infinite geometric sequence		
	Book Section 1C	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*	
The sum of infinite geometric sequences.	Use: $S_\infty = \frac{u_1}{1-r}$ to find the sum of an infinite geometric sequence.	46	☐
	Use the condition $\|r\| < 1$ to check if an infinite geometric sequence is convergent.	47	☐

H1.12	**Definitions of complex numbers**	
	Book Section 6A	Revised ☐
Syllabus wording	*You need to be able to:*	*Question*
Cartesian form $z = a + b\mathrm{i}$; the terms real part, imaginary part, conjugate, modulus and argument.	Find the real and imaginary part of complex numbers, Re(z) and Im(z).	48 ☐
	Be able to solve problems by comparing real and imaginary parts.	49 ☐
	Understand the notation z^*.	50 ☐
	Be able to apply the complex conjugate notation in problems.	51 ☐
	Understand the terms modulus, $\lvert z\rvert = \sqrt{a^2 + b^2}$ and arg z where $\tan(\arg z) = \frac{b}{a}$	52 ☐
Calculate sums, differences, products quotients by hand and with technology.	Apply the basic rules of arithmetic to complex numbers.	53 ☐
	Divide by a complex number.	54 ☐
	Use technology to perform complex number arithmetic.	55 ☐
Calculating powers of complex numbers, in Cartesian form, with technology.	Use technology to find integer powers of complex numbers.	56 ☐
The complex plane.	Use and draw Argand diagrams.	57 ☐
Complex numbers as solutions to quadratic equations of the form $ax^2 + bx + c = 0$, $a \neq 0$ with real coefficients where $b^2 - 4ac < 0$.	Use the quadratic formula when the solutions are complex.	58 ☐

H1.13	**Polar form of complex numbers**	
	Book Section 6B, 6C	Revised ☐
Syllabus wording	*You need to be able to:*	*Question*
Modulus-argument (polar form): $z = r(\cos\theta + \mathrm{i}\sin\theta) = r\,\mathrm{cis}\,\theta$	Read off the modulus and argument from the polar form.	59 ☐
Exponential form: $z = r\mathrm{e}^{\mathrm{i}\theta}$	Read off the modulus and argument from the exponential form.	60 ☐
Conversion between Cartesian, polar and exponential forms, by hand and with technology.	Convert from polar or exponential form to Cartesian form by hand.	61 ☐
	Convert from Cartesian form to polar or exponential form by hand.	62 ☐
	Convert from polar or exponential form to Cartesian form using technology.	63 ☐
	Convert from Cartesian form to polar or exponential form using technology.	64 ☐
Calculate products, quotients and integer powers in polar or exponential forms.	Find products of complex numbers in polar or exponential form.	65 ☐
	Find quotients of complex numbers in polar or exponential form.	66 ☐
	Find powers of complex numbers in polar or exponential form.	67 ☐
Adding sinusoidal functions with the same frequencies but different phase shift angles.	Be able to add functions, including voltages, by considering real or imaginary parts of complex number expressions.	68 ☐
Geometric interpretation of complex numbers.	Understand that addition and subtraction of complex numbers can be represented as vector addition and subtraction in the Argand plane.	69 ☐
	Understand that multiplication of complex numbers can be represented as a rotation and a stretch in the Argand plane.	70 ☐

H1.14	Arithmetic of matrices	
Book Section 3A, 3B, 3C	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*
Definition of a matrix: the terms element, row, column and order for $m \times n$ matrices.	Be able to understand and use the terminology to describe matrices.	71 ☐
Algebra of matrices: equality; addition; subtraction; multiplication by a scalar for $m \times n$ matrices.	Use technology to perform basic matrix arithmetic.	72 ☐
	Perform basic matrix arithmetic by hand where necessary.	73 ☐
Multiplication of matrices.	Use technology to perform matrix multiplication.	74 ☐
	Perform matrix multiplication by hand where necessary.	75 ☐
	Use matrix multiplication in solving practical problems.	76 ☐
Properties of matrix multiplication: associativity, distributivity and non-commutativity.	Understand these terms and use them appropriately.	77 ☐
Identity and zero matrices.	Use these terms and their notation.	78 ☐
Determinants and inverses of $n \times n$ matrices with technology, and by hand for 2×2 matrices.	Calculate the determinant of matrices using technology.	79 ☐
	Calculate the inverse of square matrices using technology.	80 ☐
	Calculate the determinant of a 2×2 matrix by hand using: $\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \det\boldsymbol{A} = \|\boldsymbol{A}\| = ad - bc$	81 ☐
	Calculate the inverse of a 2×2 matrix by hand using: $\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \boldsymbol{A}^{-1} = \frac{1}{\det\boldsymbol{A}} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, ad \neq bc$	82 ☐
Awareness that a system of linear equations can be written in the form $\boldsymbol{A}x = b$.	Write linear equations in this form.	83 ☐
Solution of systems of equations using inverse matrices.	Solve systems of equations using matrices.	84 ☐
	Code and decode messages using matrices.	85 ☐

H1.15	Eigenvectors and eigenvalues	
Book Section 3D	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*
Eigenvalues and eigenvectors.	Understand the definition of an eigenvalue and eigenvector.	86 ☐
Characteristic polynomial of 2×2 matrices.	Write down a characteristic polynomial for a 2×2 matrix and use it to find eigenvalues.	87 ☐
	Find eigenvectors of a 2×2 matrix.	88 ☐
Diagonalization of 2×2 matrices (restricted to the case where there are distinct real eigenvalues).	Diagonalize a 2×2 matrix.	89 ☐
Applications to powers of 2×2 matrices.	Find powers of a matrix using: $M^n = PD^nP^{-1}$ where P is the matrix of eigenvectors and D is the diagonal matrix of eigenvalues.	90 ☐
	Applications of powers of matrices to real world problems.	91 ☐

Practice questions

1 Evaluate $(3 \times 10^{40})^2 - 5 \times 10^{80}$.

2 Evaluate $3 \times 10^{n+1} - 4 \times 10^n$, leaving your answer in the form $a \times 10^k$ where $1 \leq a < 10$.

3 Evaluate $(6 \times 10^n) \div (8 \times 10^{-n})$, leaving your answer in the form $a \times 10^k$ where $1 \leq a < 10$.

4 Find the 25th term of the following arithmetic sequence:
20, 17, 14, 11 …

5 An arithmetic sequence has first term 21 and last term 1602. If the common difference is 17, how many terms are in the sequence?

6 An arithmetic sequence has 4th term 10 and 10th term 34. Find the 20th term.

7 Find the sum of the first 30 terms of the arithmetic sequence 13, 10, 7, 4 …

8 An arithmetic sequence has $u_1 = 4$, $u_{20} = 130$. Find the sum of the first 20 terms.

9 Determine the first term and common difference of an arithmetic sequence where the sum of the first n terms is given by $S_n = \sum_{r=1}^{n} (5r + 11)$.

10 Evaluate $\sum_{1}^{100}(5r + 11)$.

11 On the first day of training for a race, Ahmed runs 500 m. On each subsequent day Ahmed runs 100 m further than the day before. How far has he run in total by the end of the 28th day?

12 Juanita invests \$300 at 2.4% simple interest. How much will be in her account after 10 years?

13 A ball is dropped and the velocity (v m s^{-1}) is measured at different times (t seconds).

t	0	0.1	0.2	0.3
v	0	1.1	1.9	2.7

It is assumed that the velocity when $t = 0$ is correct, but there is uncertainty in the remaining measurements.

a By modelling the situation as an arithmetic sequence, estimate the velocity when $t = 0.5$.

b Make one criticism of the model.

14 Find the 10th term of the geometric sequence 32, –16, 8, –4 …

15 Find the number of terms in the geometric sequence 1, 2, 4 … 4096.

16 A geometric series has third term 16 and seventh term 256. Find the possible values of the first term and the common ratio.

17 Find the sum of the first eight terms of the sequence 162, 54, 18 …

18 Determine the first term and common ratio of a geometric sequence where the sum of the first n terms is given by $S_n = \sum_{r=1}^{n} 2 \times 5^r$.

19 Evaluate $\sum_{1}^{10} 2 \times 5^r$.

20 The population of bacteria in a petri dish grows by 20% each day. There are initially 50 000 bacteria in the dish.

a Find the number of bacteria in the dish after 12 days.

b Explain why this model cannot continue indefinitely.

21 £2000 is invested in an account paying 4% nominal annual interest, compounded monthly. Find the amount in the account after 10 years, giving your answer to two decimal places.

22 Samira wants to invest £1000 in an account paying a nominal annual interest rate of i%, compounded quarterly. She wants to have £1500 in her account after 8 years. What value of i is required?

23 James invests $100 in an account paying 2.1% annual interest. How many complete years are required until he has doubled his investment?

24 A car suffers from 12% annual depreciation. If the initial value is $40 000, find the value after 4 years.

25 Clint invests $2000 at 3.2% annual compound interest. He estimates that the annual inflation rate is 2.4%. Find the real value of his investment after 5 years, giving your answer to the nearest dollar.

26 Evaluate $(2^{-2})^{-2}$.

27 Simplify $(2x)^3$.

28 Solve $10^x = k$.

29 If $e^{2x-6} = 5$, find x in terms of natural logarithms.

30 Evaluate $\ln 10 + \log_{10} e$.

31 Round 0.0106 to three decimal places.

32 Round 105 070 to three significant figures.

33 x is measured as 500 000 to one significant figure. y is measured as 0.1235 correct to four significant figures. Calculate xy giving your answer to an appropriate level of accuracy.

34 If $x = 12.45$ to four significant figures, find the range of possible values of x.

35 An angle is 38°. A student estimates the answer as 45°. Find the percentage error in the student's work.

36 The side of a square is 7 cm to one significant figure. Find the maximum percentage error if the area is quoted as 49 cm^2.

37 Jenny calculates the probability of getting five heads when flipping a coin eight times and gets 2.5. Explain why Jenny's answer cannot be correct.

38 Jamie borrows $1000 at 3% interest, compounded annually. She pays back the loan over 5 years. How much should she pay at the end of each year? Give your answer to two decimal places.

39 Kumar invests $100 at the beginning of each year in an account which pays 2.3% interest at the end of the year. What is the value of his investment at the end of 10 years. Give your answer to two decimal places.

40 Heidi has $50 000 she wants to use to pay out an annuity over 30 years. If she invests the money in an account paying 4% annual interest, how much can she withdraw at the end of each year? Give your answer to the nearest dollar.

41 Solve:

$$3x + 2y + 4z = -1$$
$$x + y + z = 0$$
$$10x + 7y + 4z = 6$$

42 Find the roots of the equation $x^3 - 4x^2 + 2x + 1 = 0$.

43 Simplify $\log_{10} x^2 + \log_{10} 100x$.

44 Evaluate $8^{\frac{2}{3}}$.

45 Simplify $(4x)^{-\frac{1}{2}}$.

46 Find the sum to infinity of the geometric sequence $2, \frac{2}{3}, \frac{2}{9}\ldots$

47 Find the values of x for which the following geometric series is convergent:
$1, 2x, 4x^2\ldots$

48 Find the real part of $(1 + i)(1 - 2i)$.

49 Solve $z + \text{Im}(z) = 2 - 2\text{i}$.

50 Write down z^* if $z = -\text{i} + 4$.

51 Solve $z + 2z^* = 4 + 6\text{i}$.

52 Find the modulus and argument of $z = 1 - \sqrt{3}\text{i}$.

53 Simplify $(a + b\text{i})(a + 2b\text{i})$.

54 If a and b are real, find the real part of $\frac{1}{a + b\text{i}}$

55 If $z = 2 + \text{i}$ and $w = 1 - 2\text{i}$ use technology to evaluate $2z + wz$.

56 Evaluate $(1 + \mathrm{i})^8$ using technology.

57 If $z = 1 + \mathrm{i}$ plot z, z^* and z^2 on an Argand diagram.

58 Use the quadratic formula to solve $z^2 + 4z + 13 = 0$.

59 If $z = 3\left(\cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}\right)$ find $|z|$ and $\arg z$.

60 If $z = 5\mathrm{e}^{\frac{\pi}{4}\mathrm{i}}$ find $|z|$ and $\arg z$.

61 Write $a\mathrm{e}^{\mathrm{i}b}$ in Cartesian form.

62 If $p > 0$ write $z = p + \mathrm{i}p$ in polar form.

63 Use technology to write $2e^{\frac{i\pi}{2}}$ in Cartesian form.

64 Use technology to write $1 - \sqrt{3}i$ in polar form.

65 Write $2\operatorname{cis} A \times 5\operatorname{cis} B$ in polar form.

66 Write $\frac{10e^{6i}}{2e^{2i}}$ in exponential form.

67 If $z = 2\operatorname{cis}\theta$ find z^5 in polar form.

68 Two AV voltage sources are connected in a circuit. If $V_1 = 3\cos(100t)$ and $V_2 = 4\cos(100t + 1)$, find an expression for the total voltage in the form $V = A\cos(100t + 10)$.

69 The point 3 + 2i on the Argand diagram is translated 2 up and 4 to the right. Find the complex number of the new point.

70 Describe the transformation that maps the point z to the point $2iz$.

71 For the matrix $\begin{pmatrix} 2 & 1 \\ 5 & 0 \\ 7 & -1 \end{pmatrix}$ write down

a the order of the matrix

b the element in the first row and second column.

72 Use technology to calculate $\begin{pmatrix} 10 & 2 & 12 \\ 4 & 3 & 7 \end{pmatrix} - 3 \times \begin{pmatrix} 2 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

73 Evaluate $k\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix} - 2\begin{pmatrix} k & 1 \\ 1 & -k \end{pmatrix}$.

74 Use technology to evaluate

$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 1 \\ 2 & 3 \\ 0 & -1 \end{pmatrix}$.

75 Evaluate $\begin{pmatrix} k & 3 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 1 & -k \\ 4 & 5 \end{pmatrix}$.

76 A shop sells widgets and gizmos. The table below shows the number of widgets and gizmos sold in two different weeks:

	Widgets	**Gizmos**
Week 1	12	15
Week 2	10	18

The table below shows the profit made in $ per widget or gizmo sold:

	Widgets	**Gizmos**
Profit	0.4	0.6

Write down an appropriate matrix multiplication based on these two tables to find the total profit each week.

Did the shop make more profit on these items in the first week or the second week?

77 Show, using the matrices $\boldsymbol{A} = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}$, that matrix multiplication is non-commutative.

78 Given that $\boldsymbol{A} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, find the value of k such that $\boldsymbol{A}^2 - 3\boldsymbol{A} + k\boldsymbol{I} = 0$.

79 Find $\det\begin{pmatrix} 1 & 2 & -2 \\ 2 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$.

80 Find $\begin{pmatrix} 1 & 2 & -2 \\ 2 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}^{-1}$.

81 Find the determinant of $\begin{pmatrix} k & k^2 \\ k & 1 \end{pmatrix}$.

82 Find $\begin{pmatrix} k & k^2 \\ k & 1 \end{pmatrix}^{-1}$ given that $k > 1$.

83 The equations:

$$2x + 2y = 7$$
$$5x - y = 0$$

can be written in the form $\boldsymbol{A}x = b$.
Write down $\boldsymbol{A}$, x and b.

84 Given that $\begin{pmatrix} 1 & -3 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ k \end{pmatrix}$, use matrix methods to find x and y.

85 A rearrangement cipher takes sequences of length four and places them in a vector. The vector is then multiplied by the matrix

$$\boldsymbol{A} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

For example the word EASY is mapped to YAES since

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} E \\ A \\ S \\ Y \end{pmatrix} = \begin{pmatrix} Y \\ A \\ E \\ S \end{pmatrix}$$

If there are more than four letters, the letters are split up into consecutive four-letter chunks.

a Encode the message
HUNDREDS

b Find $\boldsymbol{A}^{-1}$.

c Hence decode
HAMTTMEALCIA

86 Show that $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ is an eigenvector of $\begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}$.

Write down the associated eigenvalue.

87 Write down the characteristic polynomial of the matrix $\begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix}$.
Hence find the eigenvalues of the matrix.

88 Use your answer to question **87** to find two non-parallel eigenvectors of the matrix $\begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix}$.

89 Use your answers to questions **87** and **88** to diagonalize the matrix $\begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix}$.

90 The matrix $\boldsymbol{M} = \begin{pmatrix} 0.9 & 0.2 \\ 0.2 & 0.6 \end{pmatrix}$ can be written as $\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0.5 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}^{-1}$.

Find the limit of M^n as $n \to \infty$.

91 The populations of adult and juvenile meerkats in a population n years after being introduced to an island are labelled A_n and J_n respectively.
A model for the population growth is given by

$$\begin{pmatrix} A_{n+1} \\ J_{n+1} \end{pmatrix} = \boldsymbol{M} \begin{pmatrix} A_n \\ J_n \end{pmatrix}$$

where $\boldsymbol{M} = \begin{pmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{pmatrix}$.

a Suggest which number parameterizes the birth rate.

b Find the eigenvalues and eigenvectors of the matrix **M**. Hence diagonalize **M**.

c Given that there are initially 1000 adults and 500 juveniles, find the long-term adult population size. Give your answer to the nearest integer.

2 Functions

Syllabus content

S2.1	**Equation of a straight line**		
	Book Section 4A	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*	
Different forms of the equation of a straight line.	Use: the gradient-intercept form $y = mx + c$; the general form $ax + by + d = 0$; the point-gradient form $y - y_1 = m(x - x_1)$ to find the equation of a straight line.	1	☐
	Find the equation of a line given its gradient and a point on the line.	2	☐
	Find the equation of a line given two points on the line. Use: $m = \frac{y_2 - y_1}{x_2 - x_1}$ for the gradient.	3	☐
Parallel lines $m_1 = m_2$.	Find the equation of a line through a given point parallel to another line.	4	☐
Perpendicular lines $m_1 \times m_2 = -1$.	Find the equation of a line through a given point perpendicular to another line.	5	☐

S2.2	**Concept of a function**		
	Book Section 3A	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*	
Function notation.	Use function notation.	6	☐
Domain, range and graph.	Find the domain of a function.	7	☐
	Find the range of a function.	8	☐
Informal concept that an inverse function reverses or undoes the effect of a function.	Understand that an inverse function reverses the effect of a function.	9	☐
Inverse function as a reflection in the line $y = x$, and the notation $f^{-1}(x)$.	Sketch the graph of the inverse function from the graph of the function.	10	☐

S2.3	**Graph of a function**		
	Book Section 3B	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*	
Creating a sketch from information given or a context.	Sketch a graph from a list of features or from a given context.	11	☐
Using technology to graph functions.	Sketch the graph of a function from a plot on the GDC.	12	☐

S2.4	**Key features of graphs**		
	Book Section 3B	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*	
Determine key features of graphs.	Use your GDC to find vertices (maximum and minimum values) and lines of symmetry.	13	☐
	Use your GDC to find vertical and horizontal asymptotes.	14	☐
	Use your GDC to find zeros of functions or roots of equations.	15	☐
Finding the point of intersection of two curves or lines using technology.	Use your GDC to find intersections of graphs.	16	☐

S2.5a	**Linear models**		
	Book Section 13A	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*	
Linear models: $f(x) = mx + c$.	Form a linear model from given data.	17	☐
	Form a piecewise linear model from two or more line segments.	18	☐

S2.5b	Quadratic models		
	Book Section 13B	Revised ☐	
Syllabus wording	*You need to be able to:*		*Question*
Quadratic models: $f(x) = ax^2 + bx + c$.	Form a quadratic model from given data.		19 ☐
Axis of symmetry, vertex, zeros and roots, intercepts on the x-axis and y-axis.	Find the axis of symmetry, vertex, zeros and y-intercept of the graph of a quadratic model. The equation for the axis of symmetry: $x = -\frac{b}{2a}$.		20 ☐

S2.5c	Exponential models		
	Book Section 13C	Revised ☐	
Syllabus wording	*You need to be able to:*		*Question*
Exponential growth and decay models: $f(x) = ka^x + c$ $f(x) = ka^{-x} + c$ $f(x) = ke^{rx} + c$	Form an exponential growth/decay model from given data.		21 ☐
Equation of a horizontal asymptote.	Find the equation of the horizontal asymptote of the graph of an exponential model.		22 ☐

S2.5d	Direct/inverse variation and cubic models		
	Book Section 13D	Revised ☐	
Syllabus wording	*You need to be able to:*		*Question*
Direct/inverse variation: $f(x) = ax^n$.	Find a direct/inverse relationship from given data.		23 ☐
The y-axis as a vertical asymptote when $n < 0$.	Sketch the graph of a function of the form $y = -\frac{a}{x^n}$.		24 ☐
Cubic models: $f(x) = ax^3 + bx^2 + cx + d$.	Form a cubic model from given data.		25 ☐

S2.5e	Sinusoidal models		
	Book Section 13E	Revised ☐	
Syllabus wording	*You need to be able to:*		*Question*
Sinusoidal models: $f(x) = a\sin(bx) + d$, $f(x) = a\cos(bx) + d$.	Find the amplitude, period and principal axis of a sinusoidal model of the form $f(x) = a\sin(bx) + d$ or $f(x) = a\cos(bx) + d$.		26 ☐

S2.6	Modelling skills		
	Book Section 13F	Revised ☐	
Syllabus wording	*You need to be able to:*		*Question*
Develop and fit the model.	Select an appropriate model based on the shape of the graph or context of the situation.		27 ☐
	Find the parameters of the chosen model from the given data.		28 ☐
	Determine a reasonable domain for a model.		29 ☐
Use the model.	Use a model to make predictions.		30 ☐
Test and reflect upon the model.	Comment on the appropriateness and reasonableness of a model.		31 ☐
	Suggest improvements to a model.		32 ☐

H2.7a	Composite functions		
	Book Section 5A	Revised ☐	
Syllabus wording	*You need to be able to*:		*Question*
Composite functions in context. The notation $(f \circ g)(x) = f(g(x))$.	Find the composite function of two functions.		33 ☐
	Find the domain of a composite function.		34 ☐

H2.7b	**Inverse functions**	
	Book Section 5B	Revised ☐
Syllabus wording	*You need to be able to*:	*Question*
Inverse function, f^{-1}, including domain restriction.	Understand that a function has to be one-to-one to have an inverse.	35 ☐
Finding an inverse function.	Find the inverse of a function.	36 ☐

H2.8	**Transformations of graphs**	
	Book Section 5C	Revised ☐
Syllabus wording	*You need to be able to:*	*Question*
Translations: $y = \mathrm{f}(x) + b$; $y = \mathrm{f}(x - a)$.	Recognize that $y = \mathrm{f}(x) + b$ is a vertical translation by b of $y = \mathrm{f}(x)$.	37 ☐
	Recognize that $y = \mathrm{f}(x - a)$ is a horizontal translation by a of $y = \mathrm{f}(x)$.	38 ☐
Reflections: in the x-axis $y = -\mathrm{f}(x)$, and in the y-axis $y = \mathrm{f}(-x)$.	Recognize that $y = -\mathrm{f}(x)$ is a reflection in the x-axis of $y = \mathrm{f}(x)$.	39 ☐
	Recognize that $y = \mathrm{f}(-x)$ is a reflection in the y-axis of $y = \mathrm{f}(x)$.	40 ☐
Vertical stretch with scale factor p: $y = p\mathrm{f}(x)$.	Recognize that $y = p\mathrm{f}(x)$ is a vertical stretch with scale factor p of $y = \mathrm{f}(x)$.	41 ☐
Horizontal stretch with scale factor $\frac{1}{q}$: $y = \mathrm{f}(qx)$.	Recognize that $y = \mathrm{f}(qx)$ is a horizontal stretch with scale factor $\frac{1}{q}$ of $y = \mathrm{f}(x)$.	42 ☐
Composite transformations.	Apply two vertical transformations to a graph.	43 ☐
	Apply one horizontal and one vertical transformation to a graph.	44 ☐

H2.9	**Further modelling**	
	Book Section 5D	Revised ☐
Syllabus wording	*You need to be able to:*	*Question*
Exponential models to calculate half life.	Form an exponential model given the half life.	45 ☐
Sinusoidal models: $\mathrm{f}(x) = a\sin(b(x - c)) + d$	Form a sinusoidal model given the amplitude, period, phase shift and central line.	46 ☐
Logistic models: $\mathrm{f}(x) = \frac{L}{1 + Ce^{-kx}}$; $L, k > 0$.	Form a logistic model given the carrying capacity, initial population and rate of growth of the population, using the logistic function: $\mathrm{f}(x) = \frac{L}{1 + Ce^{-kx}}$, $L, k > 0$.	47 ☐
Piecewise models.	Form continuous piecewise models.	48 ☐

H2.10	**Using logarithmic scales on graphs**	
	Book Section 1D	Revised ☐
Syllabus wording	*You need to be able to:*	*Question*
Scaling very large or small numbers using logarithms.	Linearize data from a relationship of the form $y = ax^n$.	49 ☐
Linearizing data using logarithms to determine if the data has an exponential or a power relationship using best fit straight lines to determine parameters.	Linearize data from a relationship of the form $y = ka^x$.	50 ☐
Interpretation of log–log and semi-log graphs.	Determine the relationship between x and y given log–log and semi-log graphs of the data.	51 ☐

Practice questions

1 Find the gradient and y-intercept of the line $3x - 4y - 5 = 0$.

2 Find the equation of the line with gradient –3 passing through the point (2, –4). Give your answer in the form $y = mx + c$.

3 Find the equation of the line passing through the points (–3, –5) and (9, 1). Give your answer in the form $ax + by + d = 0$, where a, b and d are integers.

4 Find the equation of the line through the point (1, 4) parallel to the line $y = 2x - 7$.

5 Find the equation of the line through the point (–2, 3) perpendicular to the line $y = -\frac{1}{4}x + 1$.

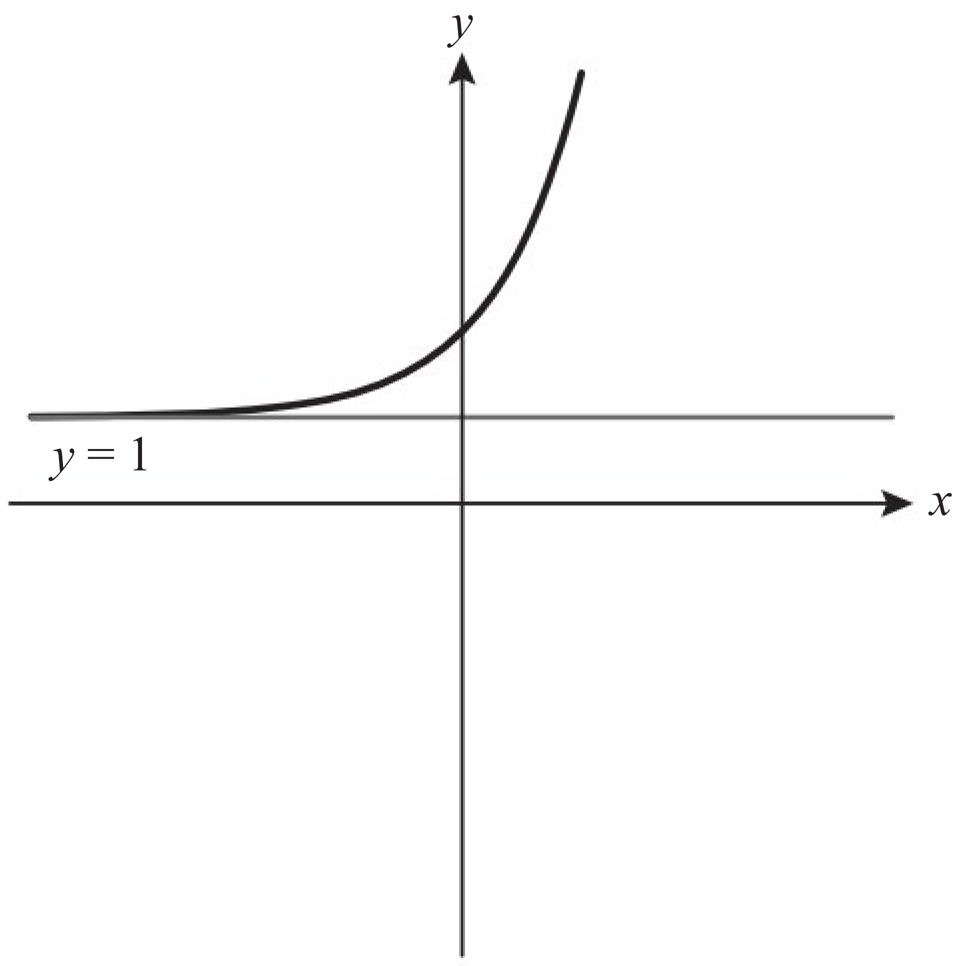
6 If $f(x) = 3x^2 - 4$, find $f(-2)$.

7 Find the largest possible domain of the function $f(x) = \ln(2x - 1)$.

8 Find the range of the function $f(x) = \sqrt{5 - x}, x \leq 1$.

9 Given that $f(x) = 4 - 3x$, find $f^{-1}(-8)$.

10 Sketch the inverse function of the following graph.



11 The graph of $y = \mathrm{f}(x)$ has zeros at –1 and 3 and no vertices. It has a vertical asymptote at $x = 1$ and a horizontal asymptote at $y = -2$.
The range of f is $\mathrm{f}(x) > -2$.
Sketch a graph with these properties.

12 Sketch the graph of $y = x^5 - x^4 + 6x^2 - 2$, labelling the y-intercept.

13 **a** Find the coordinates of the vertices of $y = x^4 + 4x^3 - 3x^2 - 14x - 8$.

b Given that the curve has a line of symmetry, find its equation.

14 Find the equation of all vertical and horizontal asymptotes of the function $f(x) = \dfrac{x^2}{x^2 + x - 6}$

15 Find the zeros of the function $f(x) = \dfrac{3}{\sqrt{x}} + 2x - 6$.

16 Find the points of intersection of $y = 3^x$ and $y = 3x + 2$.

17 Find a linear model, $f(x) = mx + c$, that satisfies $f(-5) = 10$ and $f(1) = -8$.

18 $f(x) = \begin{cases} 0.5x + 11, & 0 \le x \le 10 \\ 2x - 4, & x > 10 \end{cases}$

Find

a f(3)

b f(11).

19 Find a quadratic model, $f(x) = ax^2 + bx + c$, that satisfies $f(1) = -3$, $f(2) = 4$ and $f(3) = 17$.

20 Find a quadratic model with the following properties:

- vertex at $x = -2$
- y-intercept at $y = 9$
- passes through (1, 19).

21 Find an exponential model of the form $f(x) = k \times 3^{-x} + c$, that satisfies $f(-2) = 38$ and $f(1) = 14$.

22 $f(x) = 8e^{-0.5x} + 3$. Sketch the graph of $y = f(x)$ labelling the axis intercept and the asymptote.

23 y is inversely proportional to the square of x. Given that $y = 3$ when $x = 2$, find the relationship between x and y.

24 Sketch the graph of $y = \frac{12}{x^2}$, stating the equation of all asymptotes.

25 Find a cubic model, $f(x) = ax^3 + bx^2 + cx + d$, that satisfies $f(-2) = 1$, $f(-1) = 7$, $f(1) = 1$ and $f(2) = 13$.

26 Find a model of the form $f(x) = a\sin(bx) + d$ with the following properties:

- amplitude 4
- period 240°
- principal axis $y = -1$.

27 The following data are collected:

x	0	1	2	3	4
y	5	2.5	1.25	0.625	0.3125

Determine which one of the following models is most appropriate for the data:

$$y = ax + b$$

$$y = ax^2 + bx + c$$

$$y = \frac{a}{x}$$

$$y = k \times 2^{rx}$$

28 Find the values of the parameters of your chosen model from question **27**.

29 A publisher predicts that the percentage market share it can gain for a newly published book (s%) is given by the function $s(x) = 3x + 4$, where x (in thousands of \$) is the amount spent on marketing. Determine a suitable domain for this model.

30 From the function in question **29**, find the predicted market share when \$18 000 is spent on marketing.

31 State two reasons why the model in question **29** is unrealistic.

32 Suggest an improved model to that given in question **29** (only the form is needed; not the precise model).

33 $f(x) = \dfrac{1}{x-2}$ and $g(x) = 3x - 4$.
Find

a $f(g(x))$

b $g(f(x))$.

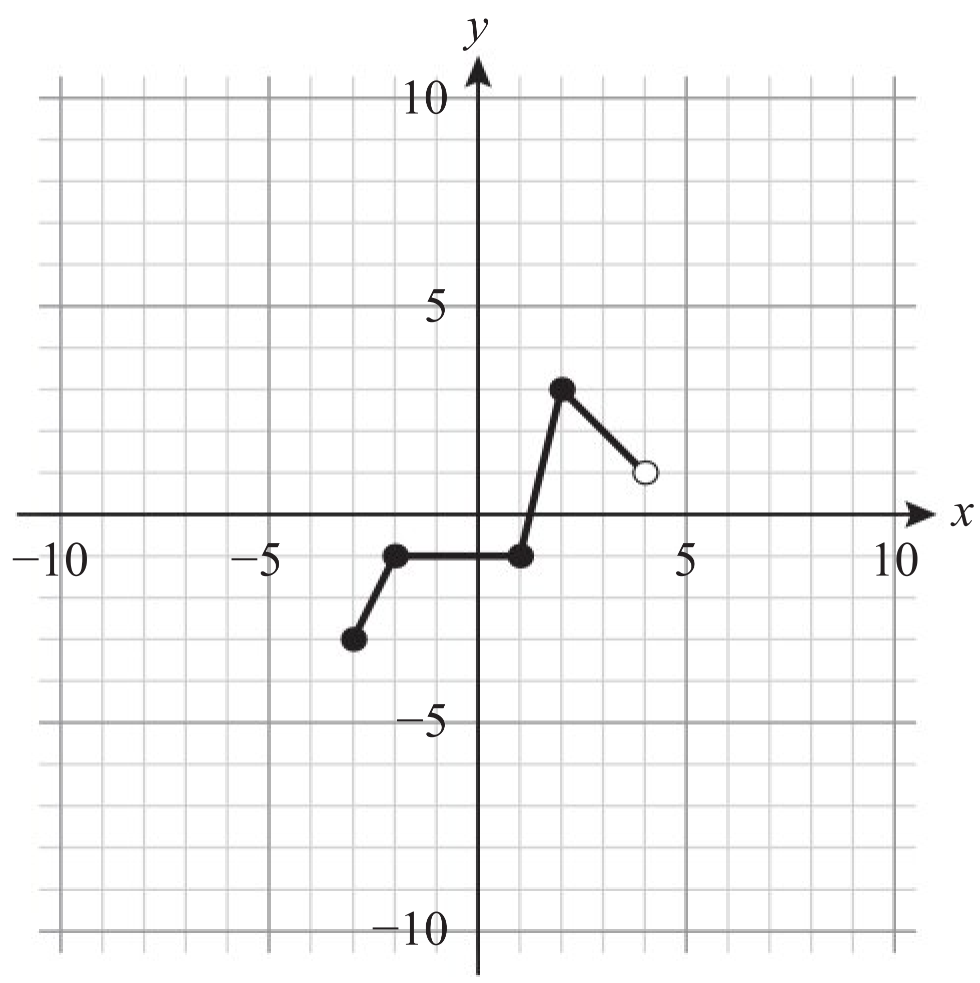
34 $f(x) = \sqrt{2-x}, x \leq 2$ and $g(x) = x - 3, x \in \mathbb{R}$.
Find the largest possible domain of $f(g(x))$.

35 Find the largest possible domain of the function $f(x) = 2xe^x, x \geq k$ for which the inverse f^{-1} exists.

36 $f(x) = \dfrac{x-1}{x+2}$

Find the inverse function $f^{-1}(x)$.

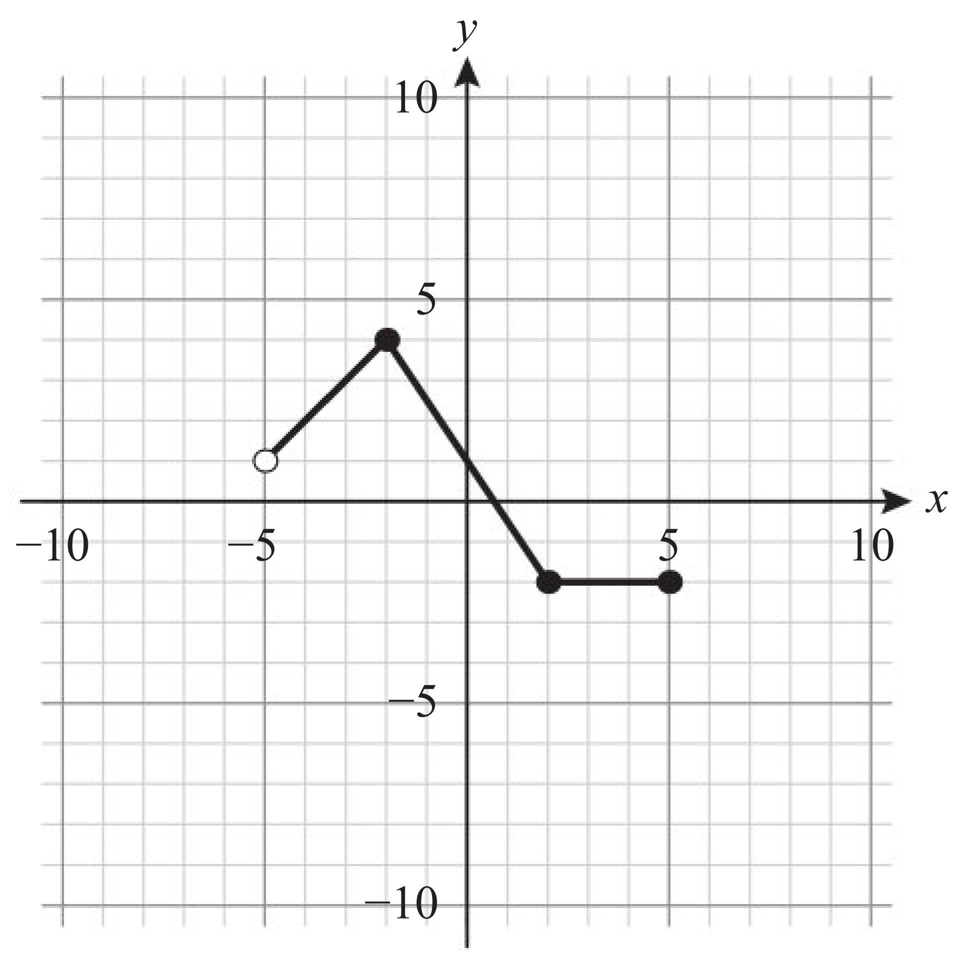
37 The graph of $y = f(x)$ is shown below.



On the axes above, sketch the graph of $y = f(x) - 4$.

38 The graph of $y = x^2 - 2x + 5$ is translated 3 units to the right. Find the equation of the resulting graph in the form $y = ax^2 + bx + c$.

39 The graph of $y = f(x)$ is shown below.

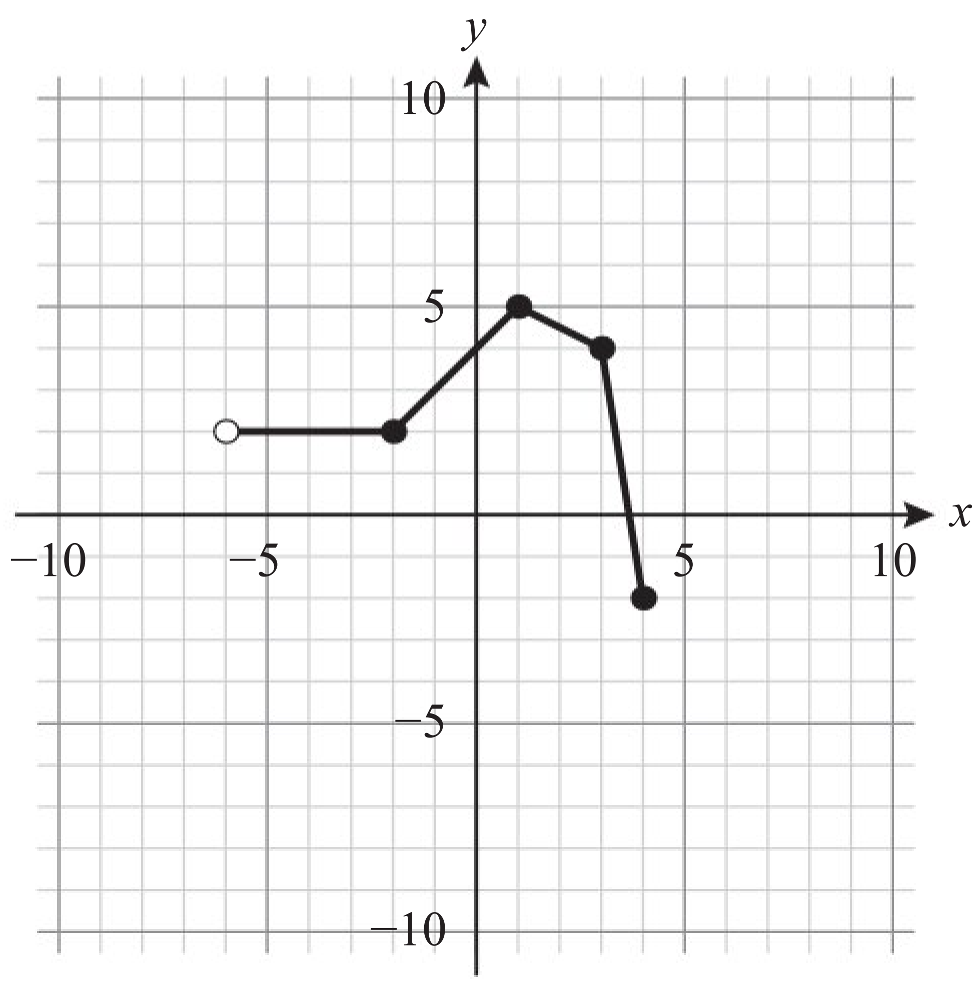


On the axes above, sketch the graph of $y = -f(x)$.

40 The graph of $y = x^3 + 3x^2 - 4x + 1$ is reflected in the y-axis.
Find the equation of the resulting graph in the form $y = ax^3 + bx^2 + cx + d$.

41 The graph of $y = 3x^2 + x - 2$ is stretched vertically with scale factor 2.
Find the equation of the resulting graph in the form $y = ax^2 + bx + c$.

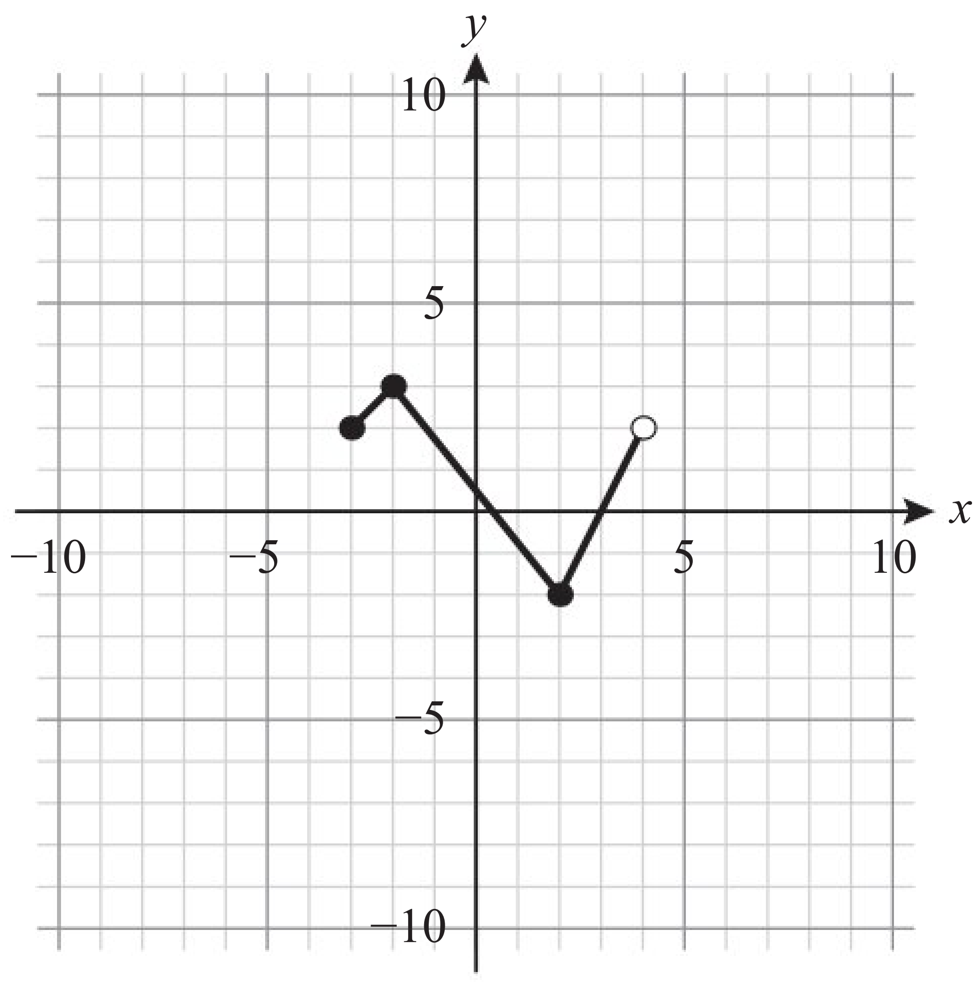
42 The graph of $y = \mathrm{f}(x)$ is shown below.



On the axes above, sketch the graph of $y = \mathrm{f}(2x)$.

43 The graph of $y = \mathrm{f}(x)$ has a single vertex at (3, –2).
Find the coordinates of the vertex on the graph $y = 4\mathrm{f}(x) + 1$.

44 The graph of $y = f(x)$ is shown below.



On the axes above, sketch the graph of $y = 3f\left(\frac{1}{2}x\right)$.

45 The background radiation level in a room is measured at 3 mSv. After a radioactive sample is introduced, the radiation level is measured at 45 mSv. If the half life of the sample is 15 minutes, form an exponential model to estimate the radiation level R after t minutes.

46 The temperature, T°C, at t hours after midnight on a summer day in a given town is modelled by a sinusoidal function with a period of 24 hours. The minimum temperature is 12°C and the maximum, which is achieved at 15:30, is 26°C.
Find a model relating T and t.

47 A farmer introduces a population of 20 sheep into an empty field with a carrying capacity of 600. After two years there are 50 sheep.
Form a logistic model for the number of sheep N after t years.

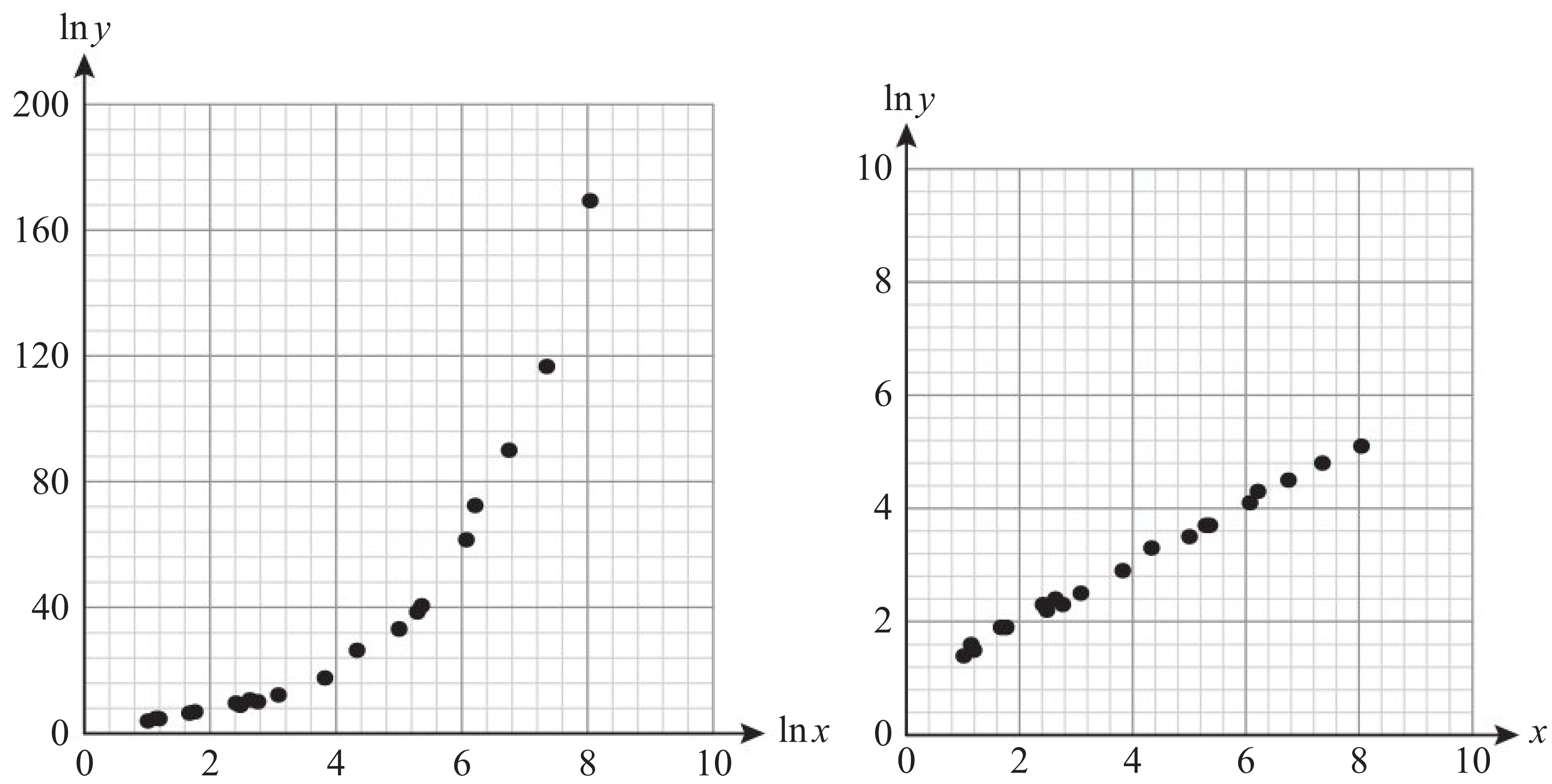
48 $f(x) = \begin{cases} x^3 - kx^2 + 1, & 0 \le x \le 3 \\ kx + 4, & x \ge 3 \end{cases}$

Given that the function f is continuous, find the value of the constant k.

49 Linearize the relationship $y = 2x^{-3}$. Describe the resulting graph.

50 Linearize the relationship $y = 5 \times 2^x$ and describe the resulting graph.

51 The log–log and semi-log graphs of a collection of data are shown below:



Determine the relationship between y and x.

3 Geometry and trigonometry

Syllabus content

S3.1a	Distance and midpoints	
	Book Section 4B	Revised ☐
Syllabus wording	*You need to be able to:*	*Question*
The distance between two points in three-dimensional space, and their midpoint.	Find the distance between points (x_1, y_1, z_1) and (x_2, y_2, z_2) using: $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$	1 ☐
	Find the midpoint using: $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$	2 ☐

S3.1b	Volume and surface area of 3D solids	
	Book Section 5A	Revised ☐
Syllabus wording	*You need to be able to:*	*Question*
Volume and area of three-dimensional solids.	Find the volume and surface area of a sphere using: $V = \frac{4}{3}\pi r^3$ $A = 4\pi r^2$ where r is the radius.	3 ☐
	Find the volume and curved surface area of a right cone using: $V = \frac{1}{3}\pi r^2 h$ $A = \pi r l$ where r is the radius, h is the height and l is the slant height.	4 ☐
	Find the volume and surface area of a right pyramid using: $V = \frac{1}{3}$ Ah where A is the area of the base and h is the height.	5 ☐
	Find the volume and surface area of combinations of solids.	6 ☐

S3.1c	Angle between intersecting lines and planes	
	Book Section 5B, 5C	Revised ☐
Syllabus wording	*You need to be able to:*	*Question*
The size of an angle between two intersecting lines or between a line and a plane.	Find the angle between two lines in two dimensions.	7 ☐
	Find the angle between a line and a plane.	8 ☐
	Find the angle between two intersecting lines in three dimensions.	9 ☐

S3.2a	Trigonometry in right-angled triangles	
	Book Section 5B	Revised ☐
Syllabus wording	*You need to be able to:*	*Question*
Use of sine, cosine and tangent ratios to find the sides and angles of right-angled triangles.	Find lengths and angles in right-angled triangles using the sine, cosine and tangent ratios.	10 ☐

S3.2b	Trigonometry in non right-angled triangles	
	Book Section 5B	Revised ☐
Syllabus wording	*You need to be able to:*	*Question*
The sine rule: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$	Find lengths and angles in non right-angled triangles using the sine rule.	11 ☐
The cosine rule: $c^2 = a^2 + b^2 - 2ab\cos C$ $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$	Find lengths and angles in non right-angled triangles using the cosine rule.	12 ☐
Area of a triangle as: $\frac{1}{2}ab\sin C$.	Find the area of a triangle when you do not know the perpendicular height.	13 ☐

S3.3	Applications of trigonometry		
	Book Section 5C	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*	
Angles of elevation and depression.	Use trigonometry in questions involving angles of elevation and depression.	14	☐
Construction of labelled diagrams from written statements.	Construct diagrams from given information (often involving bearings) and solve using trigonometry.	15	☐

S3.4	Arcs and sectors of circles		
	Book Section 14A	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*	
The circle: length of an arc; area of a sector.	Find the length of an arc: $l = \frac{\theta}{360} \times 2\pi r$ where θ is the angle measured in degrees, r is the radius.	16	☐
	Find the area of a sector: $A = \frac{\theta}{360} \times \pi r^2$ where θ is the angle measured in degrees, r is the radius.	17	☐

S3.5	Perpendicular bisectors		
	Book Section 14B	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*	
Equations of perpendicular bisectors.	Find the equation of a perpendicular bisector given the equation of a line segment and its midpoint.	18	☐
	Find the equation of a perpendicular bisector given two points.	19	☐

S3.6	Voronoi diagrams		
	Book Section 14B	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*	
Voronoi diagrams: sites, vertices, edges, cells.	Identify sites, vertices, edges and cells in a Voronoi diagram.	20	☐
Addition of a site to an existing Voronoi diagram.	Use the incremental algorithm to add a site to an existing Voronoi diagram.	21	☐
Nearest neighbour interpolation.	Use nearest neighbour interpolation to find the value of a function at any point in a Voronoi diagram.	22	☐
Applications of the toxic waste dump problem.	Solve the toxic waste dump problem to find the point which is as far as possible from any of the sites in a Voronoi diagram.	23	☐

H3.7	Radian measure		
	Book Section 4A	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*	
The definition of a radian and conversion between degrees and radians.	Convert between degrees and radians.	24	☐
Using radians to calculate area of sector, length of arc.	Find the length of an arc where the angle subtended at the centre of the circle is in radians.	25	☐
	Find the area of a sector where the angle subtended at the centre of the circle is in radians.	26	☐

H3.8	Further trigonometry		
	Book Section 4B	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*	
The definitions of $\cos\theta$ and $\sin\theta$ in terms of the unit circle.	Use the definition of $\cos\theta$ and $\sin\theta$ in terms of the unit circle.	27 ☐	
The Pythagorean identity: $\cos^2\theta + \sin^2\theta \equiv 1$.	Use the identity $\cos^2\theta + \sin^2\theta \equiv 1$.	28 ☐	
Definition of $\tan\theta$ as $\frac{\sin\theta}{\cos\theta}$.	Use the definition $\tan\theta = \frac{\sin\theta}{\cos\theta}$.	29 ☐	
Extension of the sine rule to the ambiguous case.	Use the sine rule to find two possible solutions for an angle in a triangle.	30 ☐	
Graphical methods of solving trigonometric equations in a finite interval.	Solve trigonometric equations graphically using the GDC.	31 ☐	

H3.9	Matrices as transformations		
	Book Section 4C	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*	
Geometric transformations of points in two dimensions using matrices: reflections, horizontal and vertical stretches, enlargements, translations and rotations.	Find the image of a point after reflection in the line $y = (\tan\theta)x$ using: $\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$	32 ☐	
	Find the image of a point after a horizontal stretch with scale factor k using: $\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$	33 ☐	
	Find the image of a point after a vertical stretch with scale factor k using: $\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$	34 ☐	
	Find the image of a point after an enlargement with scale factor k centre (0, 0) using: $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$	35 ☐	
	Find the image of a point after an anti-clockwise rotation of $\theta°$ using: $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$	36 ☐	
	Find the image of a point after a clockwise rotation of $\theta°$ using: $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$	37 ☐	
	Find the image of a point after a translation.	38 ☐	
Composition of the above transformations.	Find the image of a point after a composition of transformations.	39 ☐	
Geometric interpretation of the determinant of a transformation matrix.	Find the area of the image of a shape after a transformation.	40 ☐	

H3.10	Introduction to vectors		
	Book Section 2A	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*	
Representation of vectors using directed line segments.	Express vectors given as directed line segments in 2D as column vectors.	41	☐
Unit vectors; base vectors **i**, **j**, **k**.	Express column vectors in terms of the base vectors **i**, **j**, **k**.	42	☐
Components of a vector; column representation; $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$.	Express vectors given in terms of base vectors as column vectors.	43	☐
The zero vector **0**, the vector −**v**.	Add and subtract vectors algebraically and geometrically.	44	☐
	Multiply vectors by scalars.	45	☐
	Understand that two vectors are parallel when one is a scalar multiple of the other.	46	☐
Position vectors $\overrightarrow{OA} = \mathbf{a}$ $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \mathbf{b} - \mathbf{a}$	Find the displacement vector between two points with given position vectors.	47	☐
Rescaling and normalizing vectors.	Calculate the magnitude of a vector using: $\lvert\mathbf{v}\rvert = \sqrt{v_1^2 + v_2^2 + v_3^2}$	48	☐
	Find a unit vector in a given direction.	49	☐

H3.11	Vector equation of a line		
	Book Section 2B	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*	
Vector equation of a line in two and three dimensions.	Find the vector equation of a line give a point on the line and a vector parallel to the line: $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$	50	☐
$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$, where **b** is the direction vector of the line.	Find the vector equation of a line given two points on the line.	51	☐
	Convert between the vector form and the parametric form of the equation of a line: $x = x_0 + \lambda l, y = y_0 + \lambda m, z = z_0 + \lambda n$	52	☐
	Determine whether two non-parallel lines intersect.	53	☐

H3.12a	Vector applications to kinematics		
	Book Section 2C	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*	
Modelling linear motion with constant velocity in two and three dimensions.	Find the velocity vector of an object moving with constant speed.	54	☐
	Find the path of an object moving with constant velocity.	55	☐
	Determine whether two objects moving with constant velocity collide.	56	☐

H3.12b	Motion with variable velocity in 2D		
	Book Section 12B	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*	
Motion with variable velocity in two dimensions.	Differentiate or integrate accordingly to find displacement, velocity, acceleration vectors.	57	☐
	Work with vectors describing projectile motion.	58	☐
	Work with vectors describing circular motion.	59	☐

H3.13

	The scalar and vector products	
Book Section 2D	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*
Definition and calculation of the scalar product of two vectors.	Calculate the scalar product of two vectors using the definition: $\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$.	60 ☐
	Calculate the scalar product of two vectors using the definition: $\mathbf{v} \cdot \mathbf{w} = \lvert\mathbf{v}\rvert\lvert\mathbf{w}\rvert \cos\theta$ where θ is the angle between **v** and **w**.	61 ☐
The angle between two vectors; the acute angle between two lines.	Find the angle between two vectors using the result: $\cos\theta = \frac{v_1 w_1 + v_2 w_2 + v_3 w_3}{\lvert\mathbf{v}\rvert\lvert\mathbf{w}\rvert}$.	62 ☐
	Use that fact that if **v** and **w** are perpendicular then $\mathbf{v} \cdot \mathbf{w} = 0$ and if **v** and **w** are parallel then $\lvert\mathbf{v} \cdot \mathbf{w}\rvert = \lvert\mathbf{v}\rvert\lvert\mathbf{w}\rvert$.	63 ☐
	Find the angle between two lines.	64 ☐
Definition and calculation of the vector product of two vectors.	Calculate the vector product of two vectors using the definition: $\mathbf{v} \times \mathbf{w} = \begin{pmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{pmatrix}$.	65 ☐
	Calculate the magnitude of the vector product of two vectors using the definition: $\lvert\mathbf{v} \times \mathbf{w}\rvert = \lvert\mathbf{v}\rvert\lvert\mathbf{w}\rvert \sin\theta$ where θ is the angle between **v** and **w**.	66 ☐
Geometric interpretation of $\lvert\mathbf{v} \times \mathbf{w}\rvert$.	Calculate the area of a parallelogram with adjacent sides **v** and **w** using the formula: $A = \lvert\mathbf{v} \times \mathbf{w}\rvert$.	67 ☐
Components of vectors.	Find the component of vector **a** acting in the direction of vector **b** and the component acting perpendicular to vector **b**.	68 ☐

H3.14

	Introduction to graph theory	
Book Section 7A	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*
Graphs, vertices, edges, adjacent vertices, adjacent edges. Degree of a vertex.	Identify adjacent edges and vertices, and find the degree of each vertex.	69 ☐
Simple graphs; complete graphs; weighted graphs.	Understand the terms simple, complete and weighted graph.	70 ☐
	Identify whether a graph is connected or strongly connected.	71 ☐
Directed graphs; in degree and out degree of a directed graph.	Identify in degree and out degree of a directed graph.	72 ☐
Subgraphs; trees.	Understand that a tree is a graph with no cycles.	73 ☐

H3.15a

	Adjacency matrices	
Book Section 7A	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*
Adjacency matrices.	Write down an adjacency matrix for a graph.	74 ☐
	Draw a graph with a given adjacency matrix.	75 ☐

H3.15b

	Using adjacency matrices	
Book Section 7B	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*
Number of k-length walks (or less than k-length walks) between two vertices.	Use the adjacency matrix to find the number of k-length walks connecting two given vertices.	76 ☐

H3.15c	Weighted graphs		
	Book Section 7C	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*	
Weighted adjacency tables.	Represent a weighted graph by a weighted adjacency table.	77	☐
Construction of the transition matrix for a strongly connected, undirected or directed graph.	Construct a transition matrix for a given graph, and use it to find the probability of a random walk ending up at a certain vertex.	78	☐
	Use the PageRank algorithm.	79	☐

H3.16a	Moving around a graph		
	Book Section 7B	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*	
Walks, trails, paths, circuits, cycles.	Recognize different types of walk.	80	☐
Eulerian trails and circuits.	Determine whether an Eulerian trail or circuit exists by considering the degrees of the vertices.	81	☐
Hamiltonian paths and cycles.	Find a Hamiltonian path or cycle in a graph.	82	☐

H3.16b	Minimum spanning tree algorithms		
	Book Section 7D	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*	
Kruskal's and Prim's algorithms for finding minimum spanning trees.	Use Kruskal's algorithm on a graph presented by a diagram or as a table.	83	☐
	Use Prim's algorithm on a graph.	84	☐
	Use the matrix method for Prim's algorithm.	85	☐

H3.16c	Chinese postman problem		
	Book Section 7E	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*	
Chinese postman problem and algorithm for solution, to determine the shortest route around a weighted graph with up to four odd vertices, going along each edge at least once.	Perform the algorithm on a graph with two odd vertices.	86	☐
	Perform the algorithm on a graph with four odd vertices.	87	☐

H3.16d	Travelling salesman problem		
	Book Section 7F	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*	
Travelling salesman problem to determine the Hamiltonian cycle of least weight in a weighted complete graph.	Complete a table of least distances before applying the algorithm.	88	☐
Nearest neighbour algorithm for determining an upper bound for the travelling salesman problem.	Use the nearest neighbour algorithm at each step and then complete the cycle.	89	☐
Deleted vertex algorithm for determining a lower bound for the travelling salesman problem.	Find the minimum spanning tree for the remaining vertices, then add back the deleted vertex.	90	☐
	Understand that the length of the minimal cycle lies between the upper and lower bounds.	91	☐

Practice questions

1 Find the distance between (2, –4, 5) and (7, 3, –1).

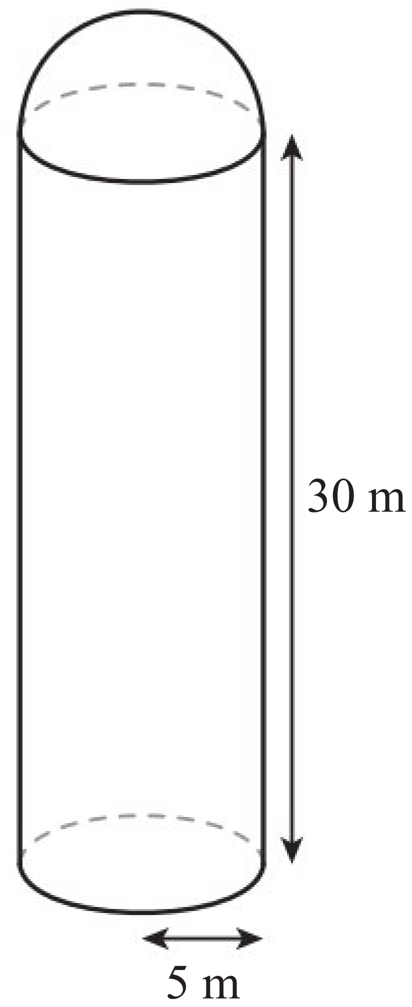
2 Find the midpoint of (1, 8, –3) and (–5, 2, 4).

3 Find, to three significant figures, the volume and surface area of a sphere of diameter 16 cm.

4 Find, to three significant figures, the volume and surface area of a cone with base radius 6 cm and height 15 cm.

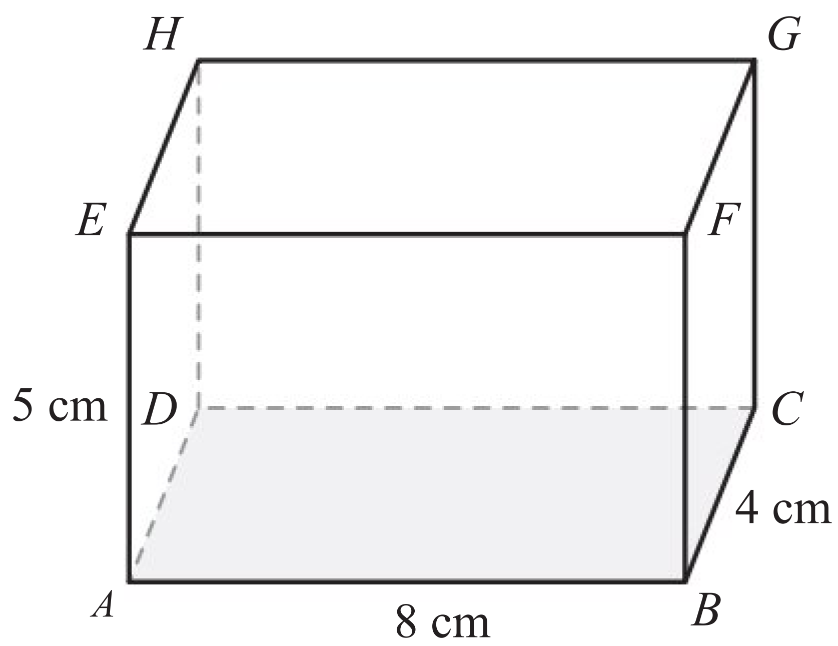
5 Find, to three significant figures, the volume and surface area of a square-based pyramid with base side 5 cm and height 9 cm.

6 A grain silo is formed of a hemisphere on top of a cylinder of radius 5 m and height 30 m as shown. Find the silo's volume.



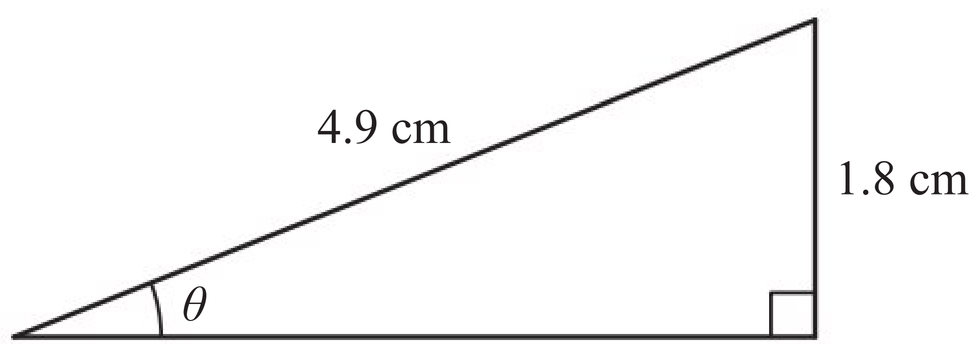
7 Find the acute angle between the lines $y = 4x - 3$ and $y = 5 - 3x$.

8 Find the angle between the line *AG* and the base plane *ABCD* in the cuboid below.

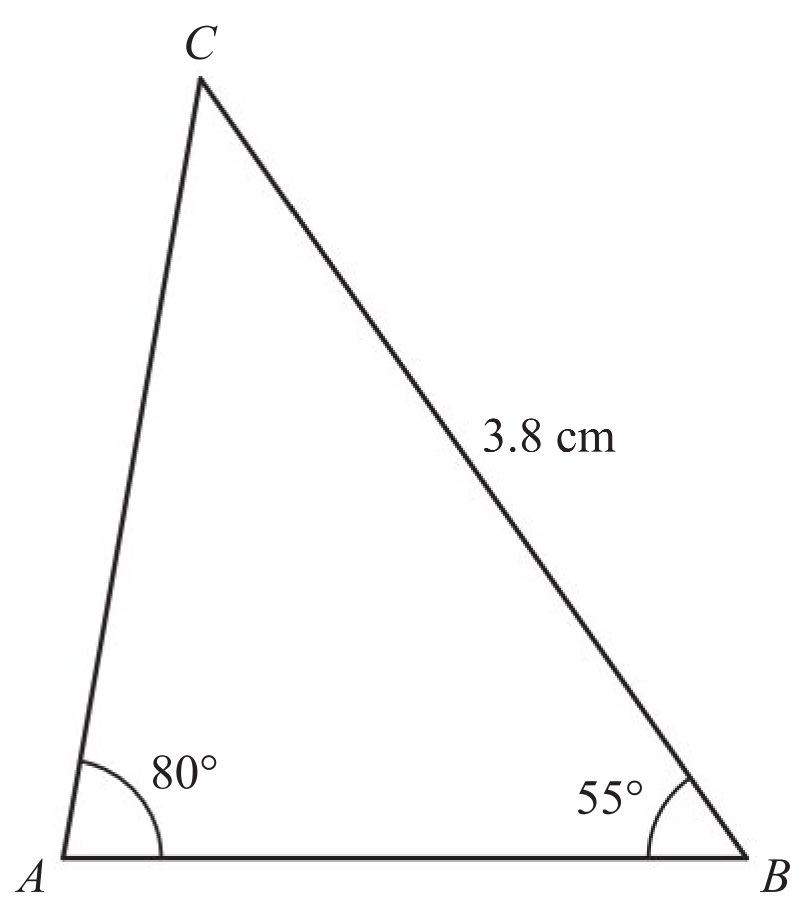


9 Find the acute angle between the diagonals *AG* and *EC* in the cuboid from question **8**.

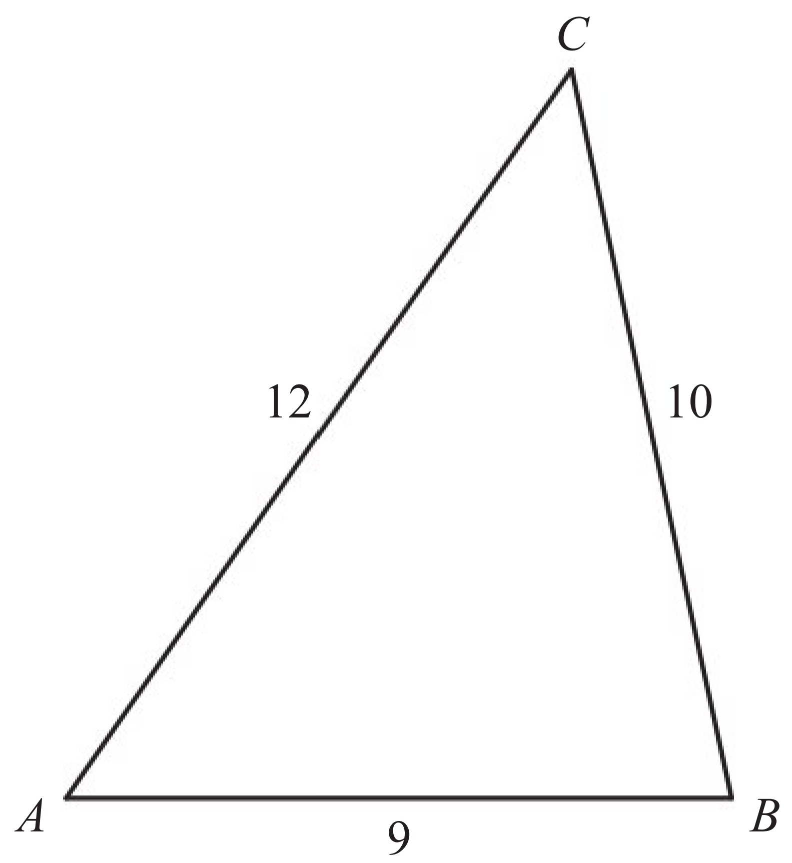
10 Find the angle θ in the following triangle.



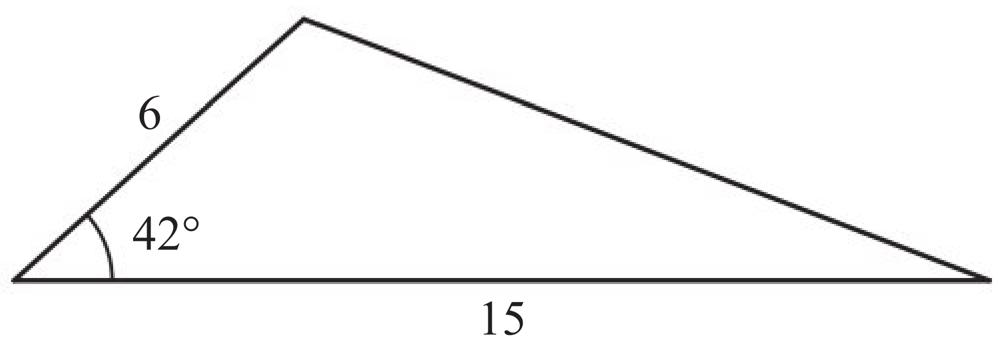
11 Find the length *AC* in the following triangle.



12 Find the angle C in the following triangle.



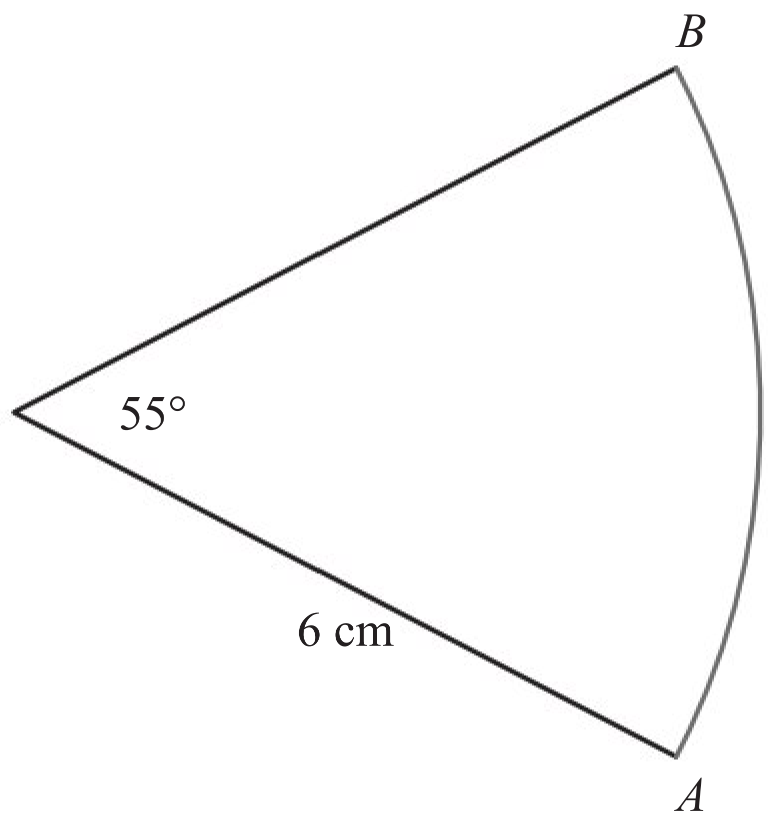
13 Find the area of the following triangle.



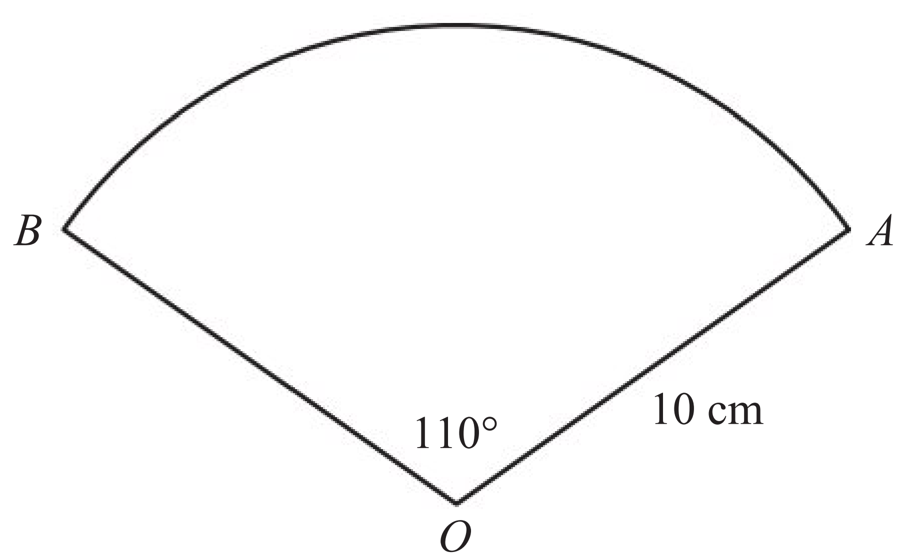
14 The angle of elevation of the top of a tree at a distance of 6.5 m is 68°. Find the height of the tree.

15 A ship leaves port on a bearing of 030° and travels 150 km before docking. It then travels on a bearing of 110° for 80 km before docking again.
Find the distance the ship must now travel to return to where it started.

16 Find the length of the arc *AB*.



17 Find the area of the sector *AOB*.



18 The line segment AB has equation $2x + 3y = 5$ and midpoint (4, 7).
Find the equation of the perpendicular bisector of AB.

19 Find the equation of the perpendicular bisector of (–3, –2) and (1, 8).

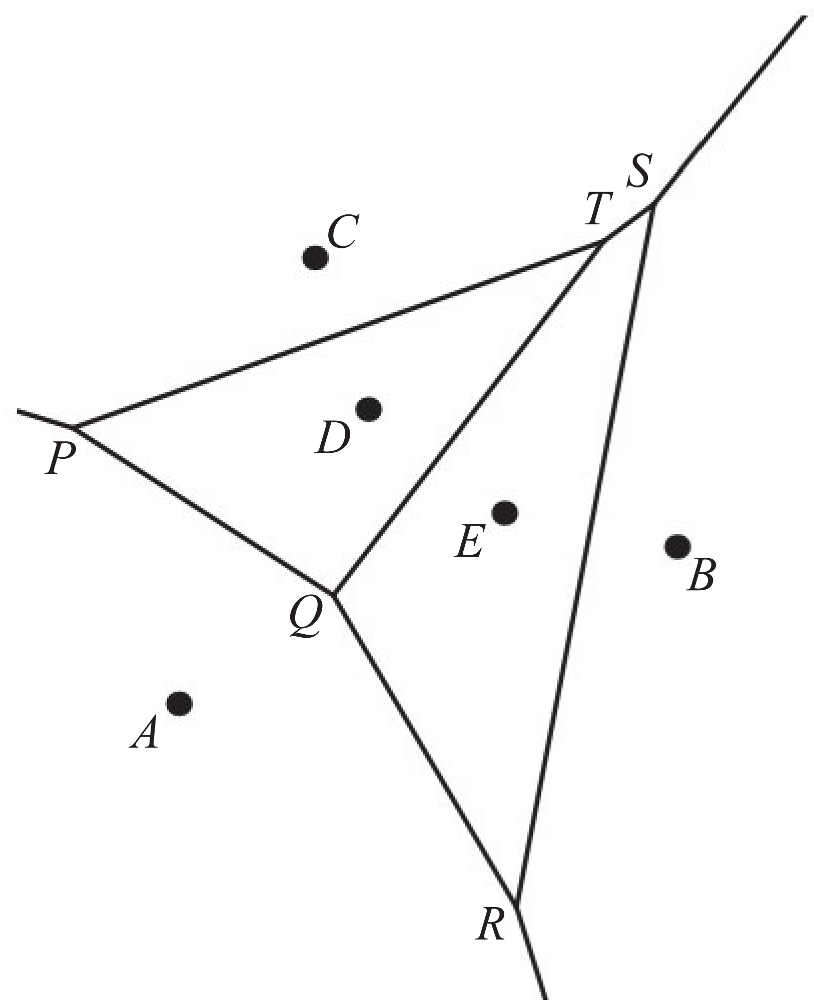
20 For the Voronoi diagram shown, identify all the:

a sites

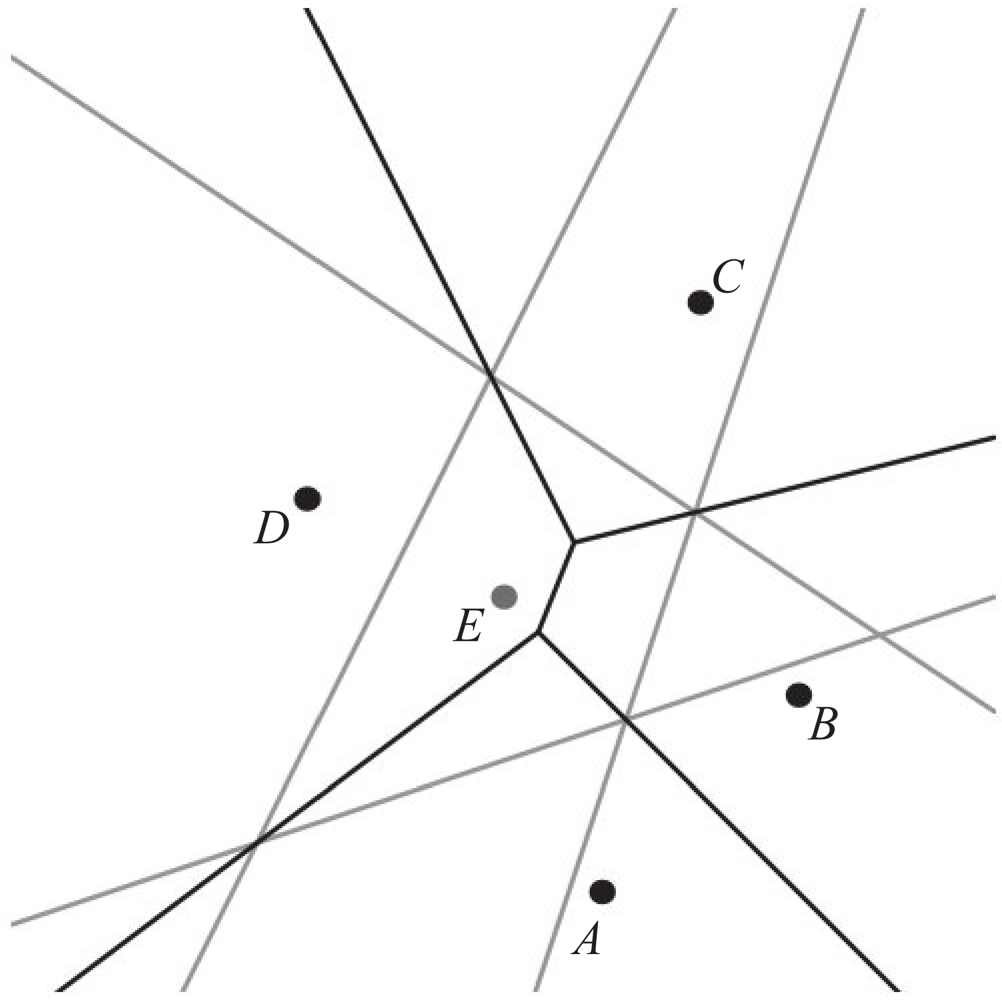
b vertices

c finite edges

d finite cells.



21 The Voronoi diagram for sites *A*, *B*, *C* and *D* is shown. An additional site *E* is added and is shown together with the perpendicular bisectors of line segments joining *E* to the other sites.

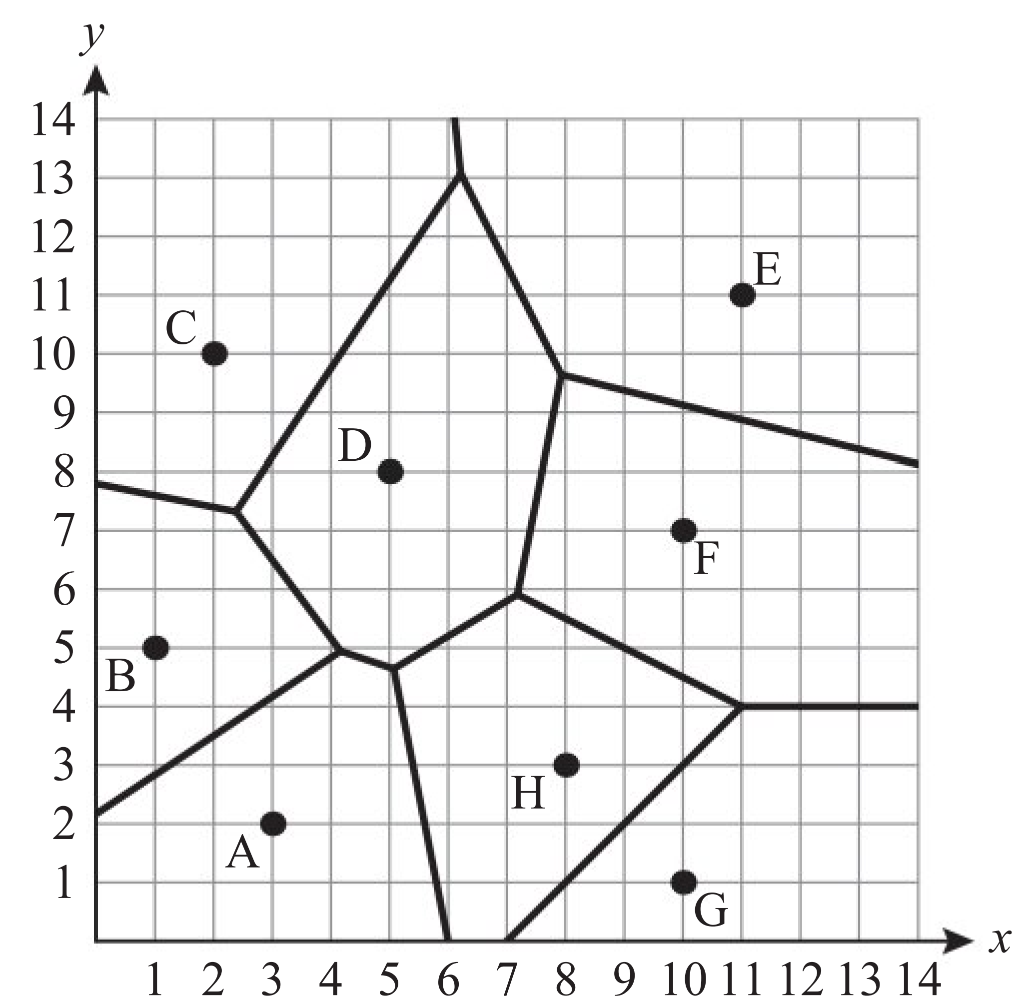


Find the Voronoi diagram for *A*, *B*, *C*, *D* and *E*.

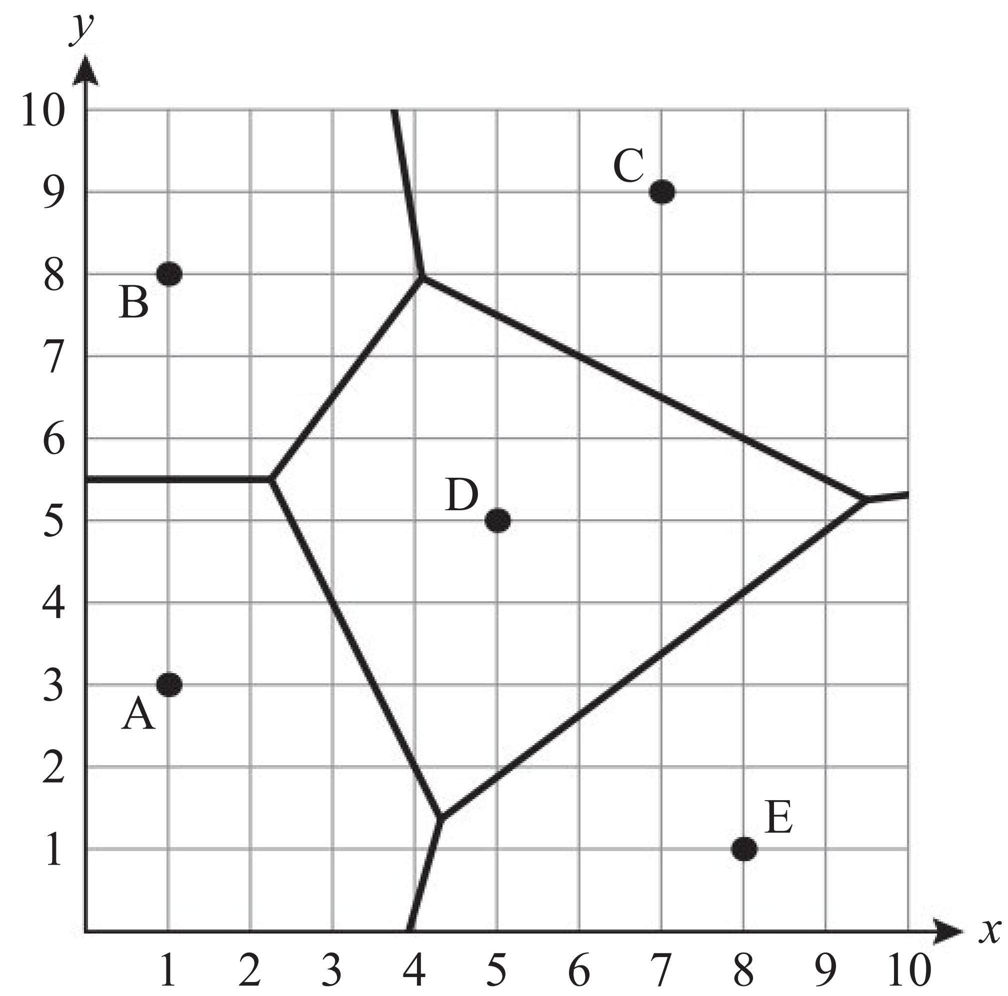
22 The function f has the following values at each site:

Site	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*
f	12	10	6	9	11	15	7	13

Use nearest neighbour interpolation to estimate the value of the function at (5, 4).



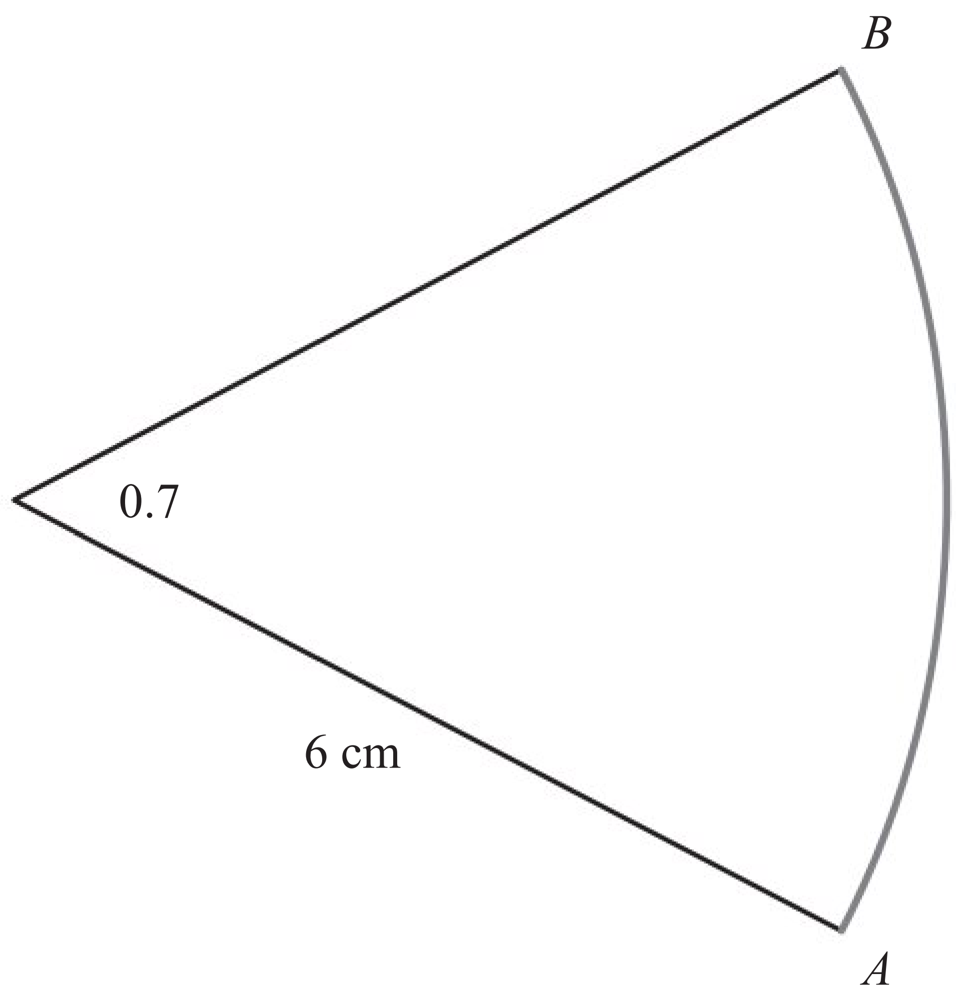
23 Towns are located at the points *A*, *B*, *C*, *D* and *E*; one unit on the graph represents 1 mile. The local authorities need to locate a toxic waste dump in this area, within 5 miles of town *D* but as far as possible from any of the towns. Find the coordinates of the required location.



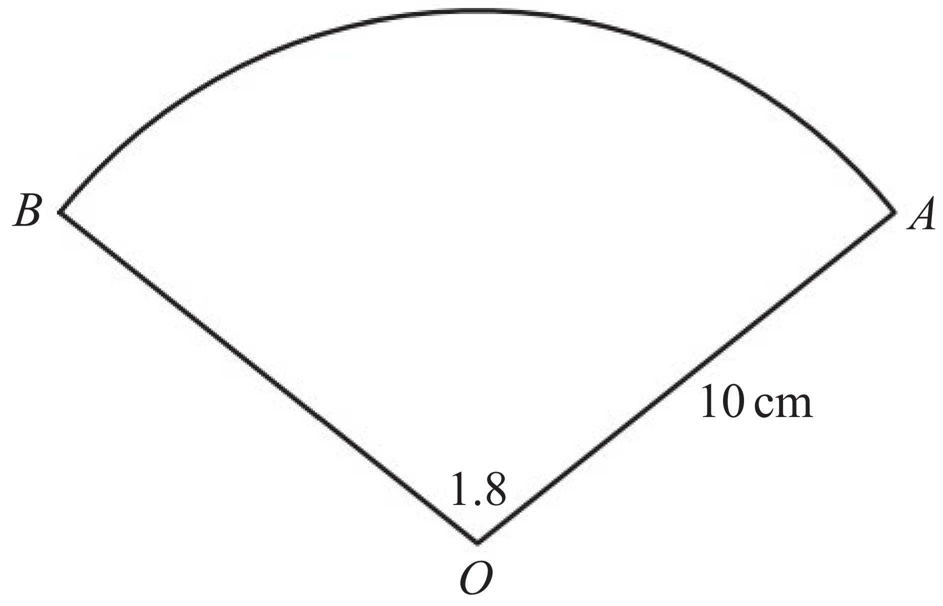
24 **a** Convert 55° to radians.

b Convert 1.2 radians to degrees.

25 Find the length of the arc *AB*.



26 Find the area of the sector *AOB*.



27 Given that $\sin \theta = 0.4$, find the value of:

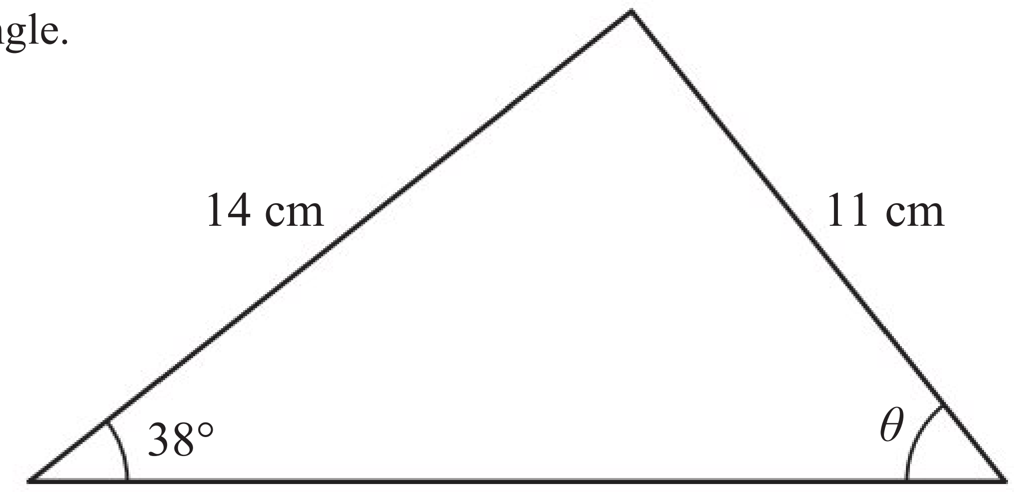
a $\sin(\theta + \pi)$

b $\cos\left(\theta - \frac{\pi}{2}\right)$.

28 Given that $\sin \theta = \frac{3}{4}$, where $\frac{\pi}{2} < \theta < \pi$, find the exact value of $\cos \theta$.

29 Using the definition of the tangent function, show that $\tan(2\pi - \theta) = -\tan \theta$.

30 Find the size of the angle θ in the following triangle.



31 Solve the equation $7 \cos\left(2x - \frac{\pi}{5}\right) = 4$ for $0 < x < \pi$.

32 Find the image of the point (3, –2) after reflection in the line $y = \frac{1}{2}x$.

33 Find the image of the point (12, 5) after a horizontal stretch with scale factor $\frac{1}{3}$.

34 Find the image of the point (–1, 4) after a vertical stretch with scale factor 3.8.

35 Find the image of the point (3, –10) after an enlargement by scale factor 2.5 about the origin.

36 Find the image of the point (4, –6) after an anti-clockwise rotation of 60° about the origin.

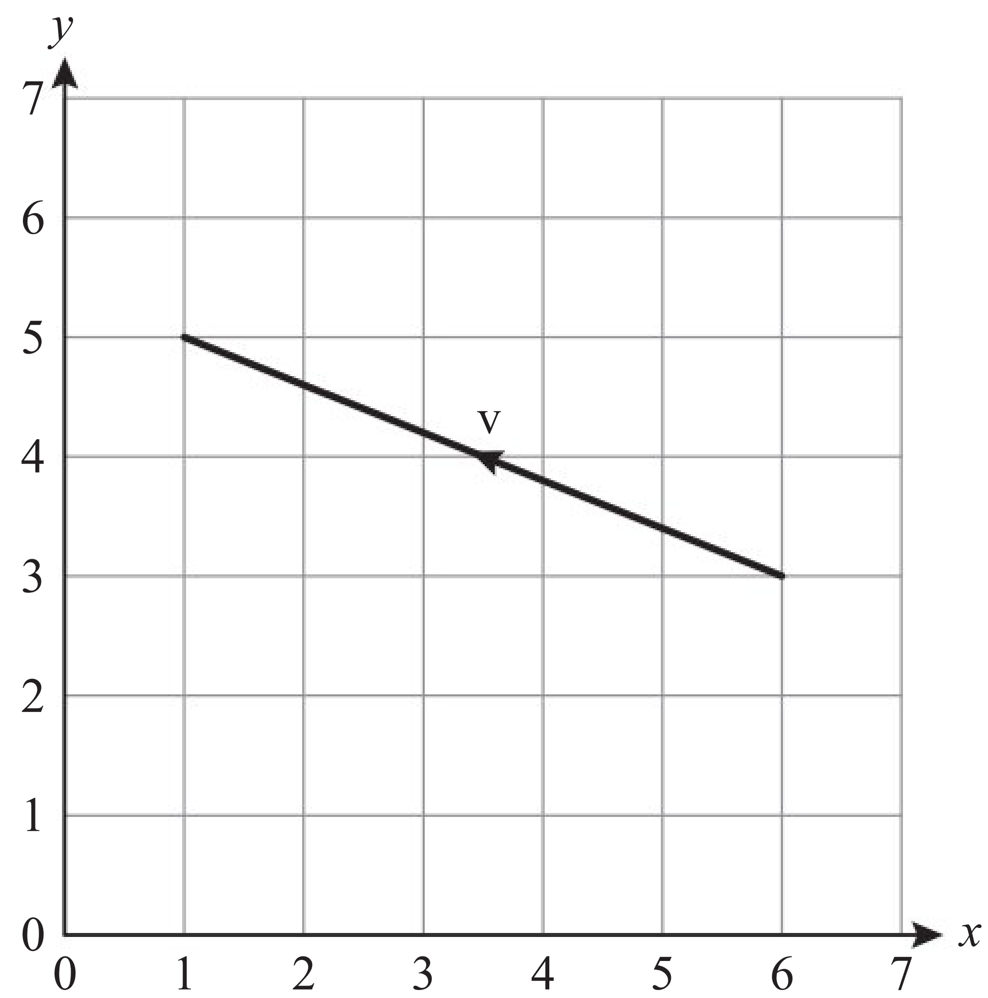
37 Find the image of the point (2, 8) after a clockwise rotation of 45° about the origin.

38 Find the image of the point (–9, 5) after a translation by $\begin{pmatrix} 6 \\ -2 \end{pmatrix}$.

39 Find the image of the point (3, 7) after a reflection in the line $y = -x$ followed by a rotation 90° anti-clockwise about the origin.

40 The triangle with vertices (2, 3), (2, 11) and (4, 6) is transformed by the matrix $\begin{pmatrix} -3 & 1 \\ 4 & 2 \end{pmatrix}$.
Find the area of the resulting triangle.

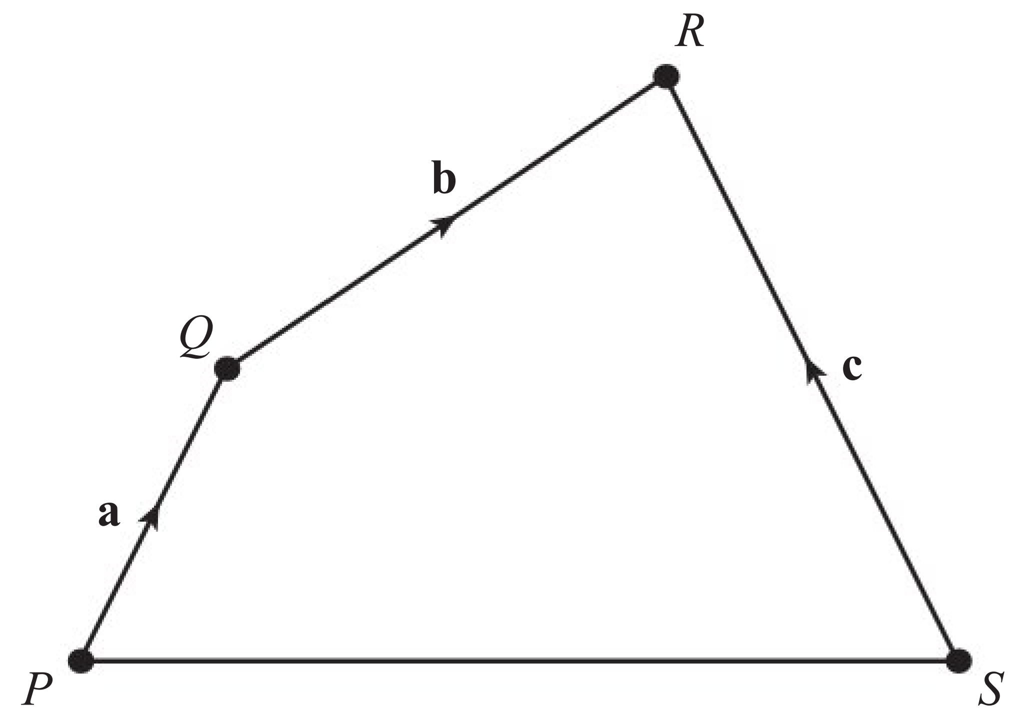
41 Express the following as a column vector:



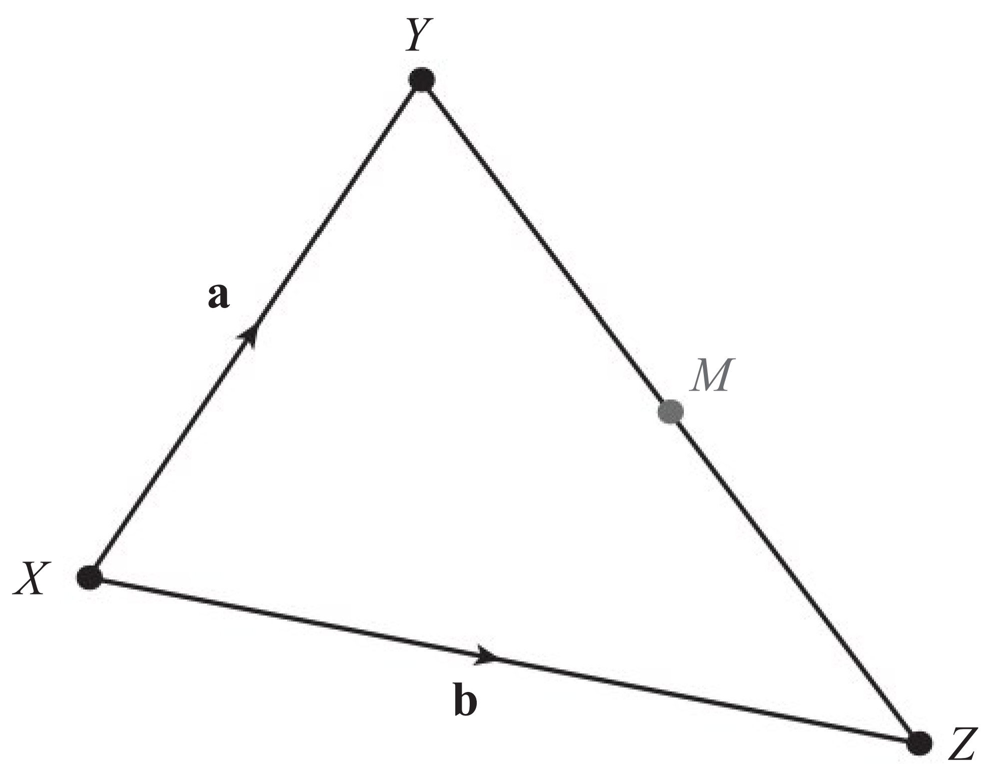
42 Express the vector $\begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix}$ in terms of base vectors.

43 Express the vector $\mathbf{i} - 6\mathbf{k}$ as a column vector.

44 Express $\overrightarrow{PS}$ in terms of the vectors **a**, **b** and **c**.



45 M is the midpoint of YZ.
Find an expression for $\overrightarrow{XM}$ in terms of the vectors **a** and **b**.



46 Given that $\mathbf{a} = \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} p \\ q \\ -12 \end{pmatrix}$ and that **a** and **b** are parallel, find the values of p and q.

47 Points A and B have coordinates (5, 1, –2) and (–4, 3, –1).
Find the displacement vector $\overrightarrow{AB}$.

48 Find the magnitude of the vector $\mathbf{v} = 3\mathbf{i} - 5\mathbf{j} - \mathbf{k}$.

49 Find a unit vector in the direction of $\begin{pmatrix} 2 \\ 6 \\ -3 \end{pmatrix}$.

50 Find the vector equation of the line parallel to the vector $\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$ passing through the point (5, 4, –7).

51 Find the vector equation of the line through the points $A(-2, 6, 1)$ and $B(3, -5, 4)$.

52 Find the parametric equation of the line with vector equation $\mathbf{r} = \begin{pmatrix} -3 \\ 0 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}$.

53 l_1: $\mathbf{r} = \begin{pmatrix} -1 \\ -7 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 1 \\ 4 \end{pmatrix}$

l_2: $\mathbf{r} = \begin{pmatrix} 2 \\ -3 \\ -6 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

Show that the lines l_1 and l_2 intersect and find the coordinates of their point of intersection.

54 Find the velocity vector of an object moving with speed $32.5\,\mathrm{m\,s^{-1}}$ in the direction of the vector $\begin{pmatrix} 3 \\ -4 \\ 12 \end{pmatrix}$.

55 An object moves with constant velocity $(3\mathbf{i} - \mathbf{j} + 2\mathbf{k})\,\mathrm{m\,s^{-1}}$. Initially it is at the point with position vector $(-5\mathbf{i} + \mathbf{k})\,\mathrm{m}$.
Find the position vector of the object after 10 seconds.

56 The position vectors of two drones at time t hours are given by

$$\mathbf{r}_A = \begin{pmatrix} 3 \\ -5 \\ 1 \end{pmatrix} + t\begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix} \text{ and } \mathbf{r}_B = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + t\begin{pmatrix} 5 \\ -8 \\ 1 \end{pmatrix}.$$

Show that the drones do not collide.

57 The velocity of an object is given by

$$\mathbf{v} = \begin{pmatrix} 3e^{t} \\ 4e^{-2t} \end{pmatrix}.$$

Initially the object is at the origin.

Find an expression for the object's:

a acceleration

b displacement.

58 The position vector of a projectile at time t is given by

$\mathbf{r} = 12t\mathbf{i} + (15t - 4.9t^2)\mathbf{j}$ m.

Find the maximum height reached.

59 The position vector of an object at time t is given by

$\mathbf{r} = \begin{pmatrix} 2\cos 3t + 1 \\ 2\sin 3t - 5 \end{pmatrix}$.

a Find the distance of the object from the point (1, −5) at time t.

b Hence describe the path of the object.

60 Given that $\mathbf{a} = \begin{pmatrix} -4 \\ 3 \\ 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ 5 \\ -3 \end{pmatrix}$, calculate $\mathbf{a} \cdot \mathbf{b}$.

61 The vectors **a** and **b** are such that $|\mathbf{a}| = 3$, $|\mathbf{b}| = 8$ and the acute angle between **a** and **b** is 45°. Find the value of $\mathbf{a} \cdot \mathbf{b}$.

62 Find the acute angle between the vectors $\mathbf{a} = \mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$ and $\mathbf{b} = 4\mathbf{i} - 2\mathbf{j} - 7\mathbf{k}$.

63 Find the value of t such that the vectors $\begin{pmatrix} 2+t \\ -3 \\ t \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 4t-1 \\ 5 \end{pmatrix}$ are perpendicular.

64 Find the acute angle between the lines with equations

$\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} -4 \\ 1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 3 \\ -5 \end{pmatrix}$.

65 Find $\begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} \times \begin{pmatrix} 5 \\ 2 \\ -2 \end{pmatrix}$.

66 The vectors **a** and **b** are such that $|\mathbf{a}| = 4$, $|\mathbf{b}| = 5$ and the acute angle between **a** and **b** is 30°. Find the magnitude of $\mathbf{a} \times \mathbf{b}$.

67 Find the area of the triangle with vertices $P(-1, 3, -2)$, $Q(4, -2, 5)$ and $R(1, 0, 3)$.

68 Find the component of the vector $10\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$ acting:

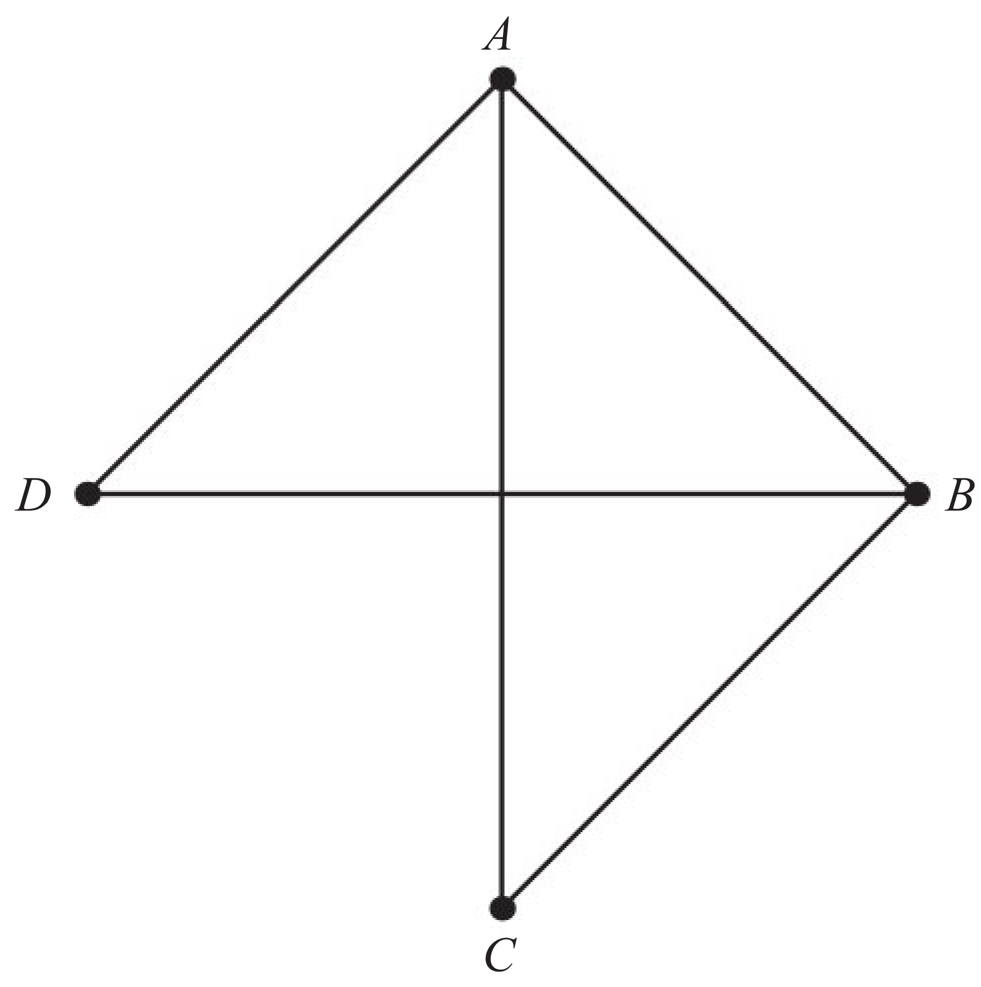
a in the direction of the vector $2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$

b perpendicular to the vector $2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$.

69 For the graph below:

a write down all the vertices adjacent to B

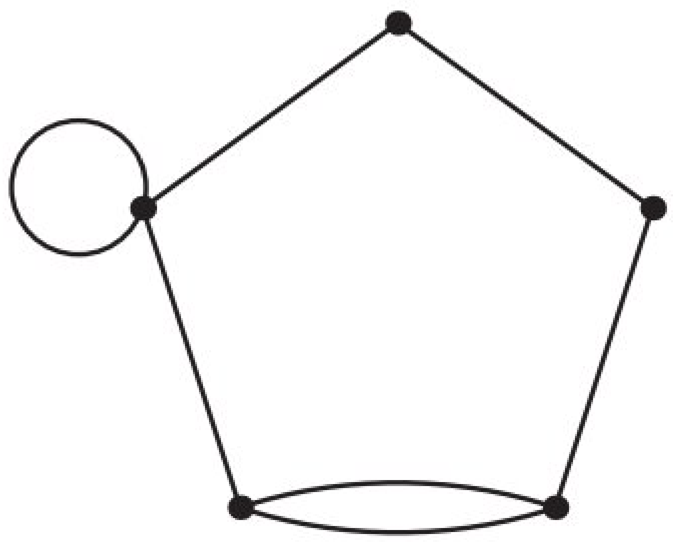
b write down the degree of vertex C.



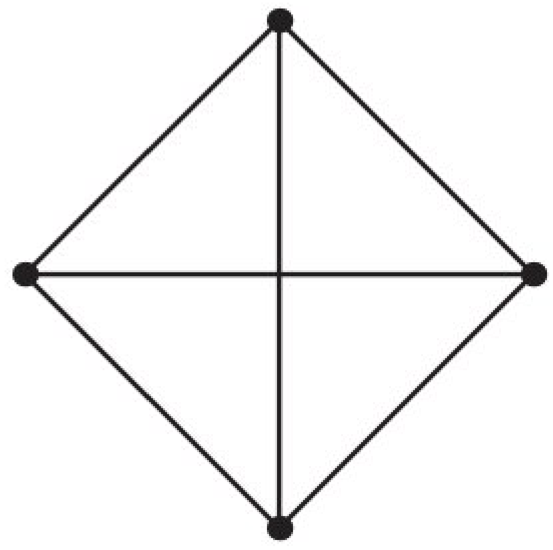
70 State which of the graphs below are:

a simple

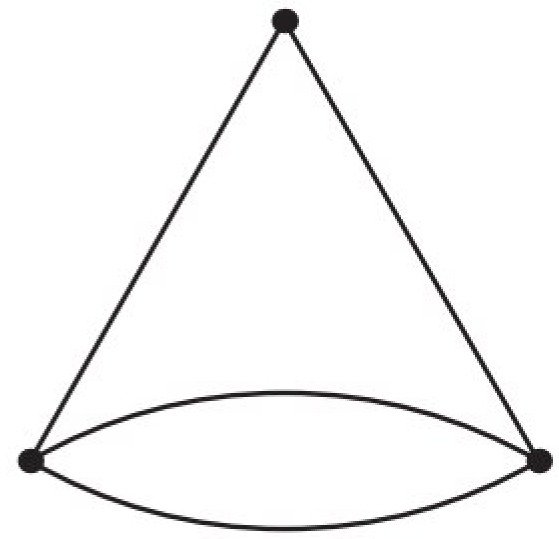
b complete.



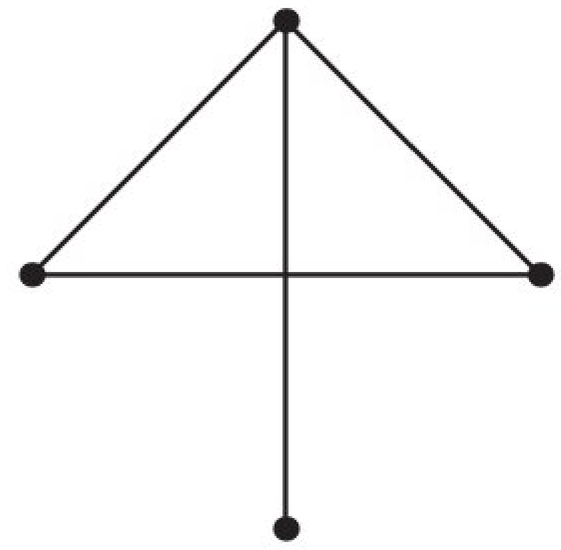
A



B

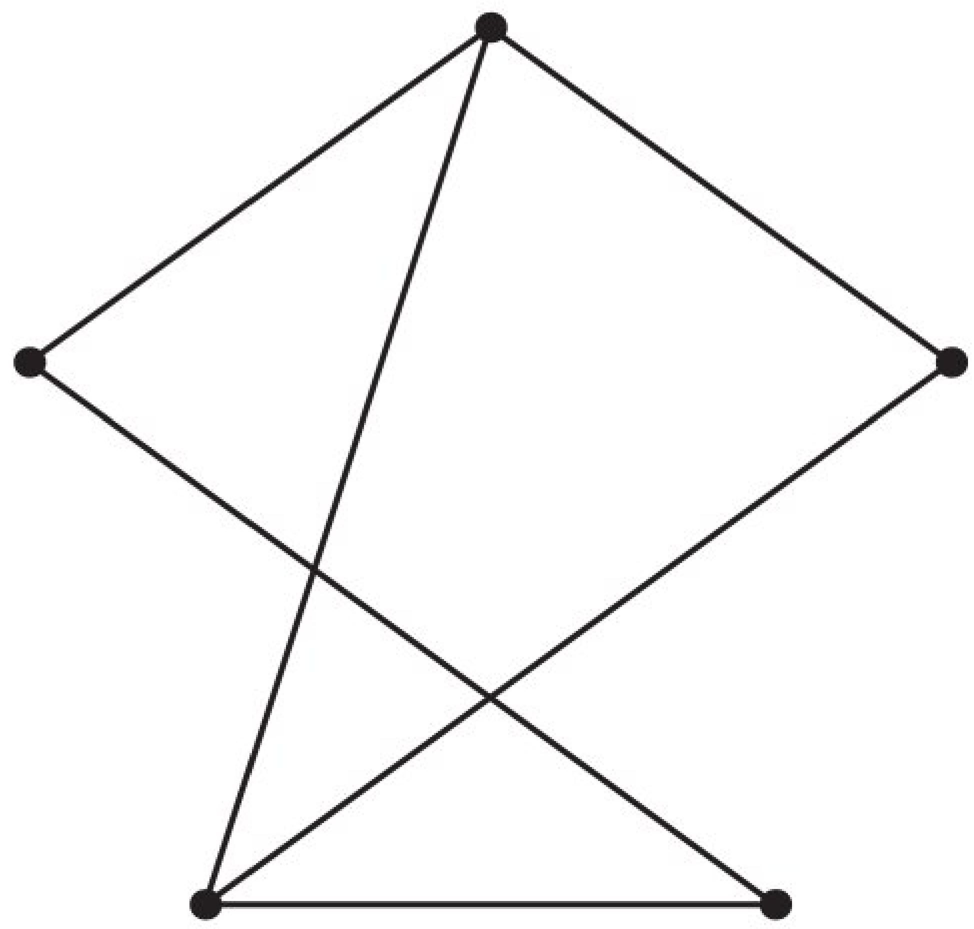


C

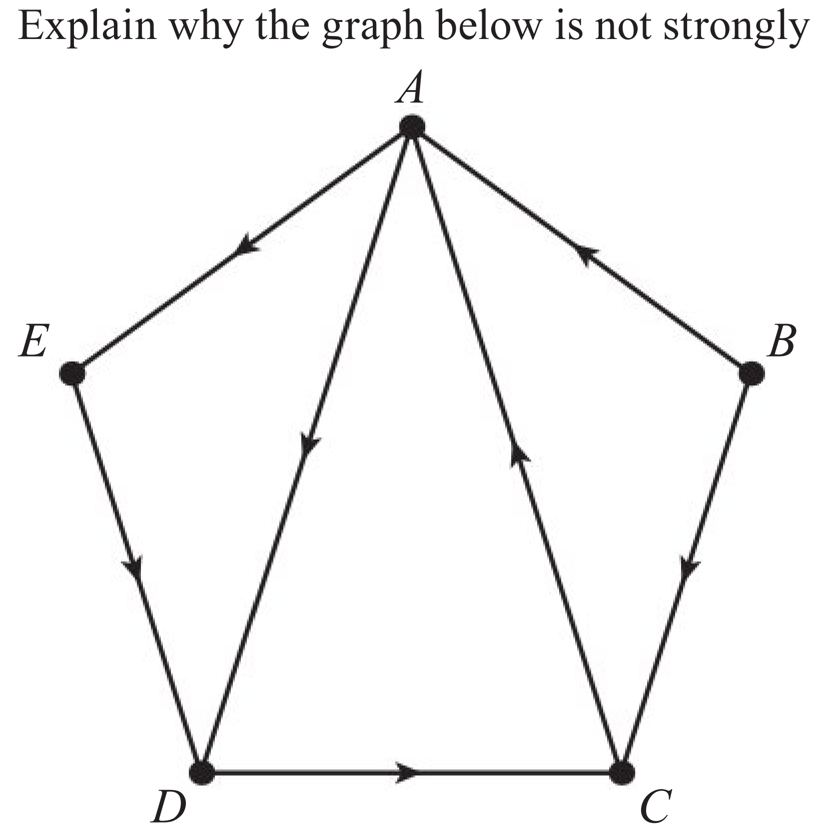


D

71 a Is the graph shown below connected?



b Explain why the graph below is not strongly connected.

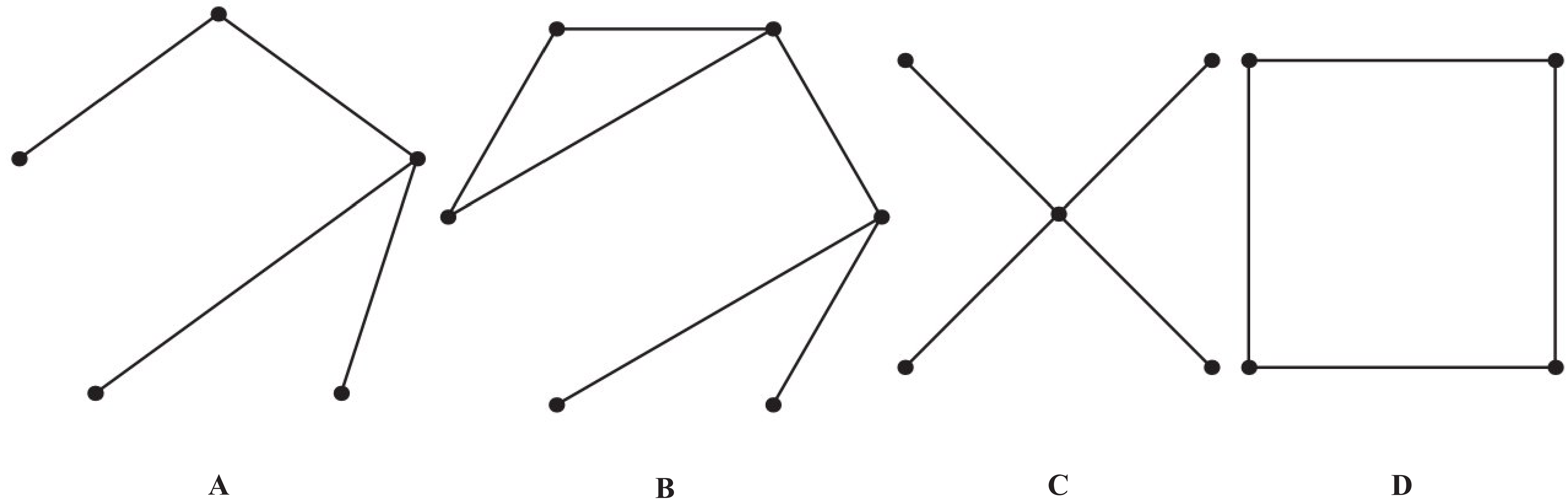


72 For the graph in question **71b**, identify:

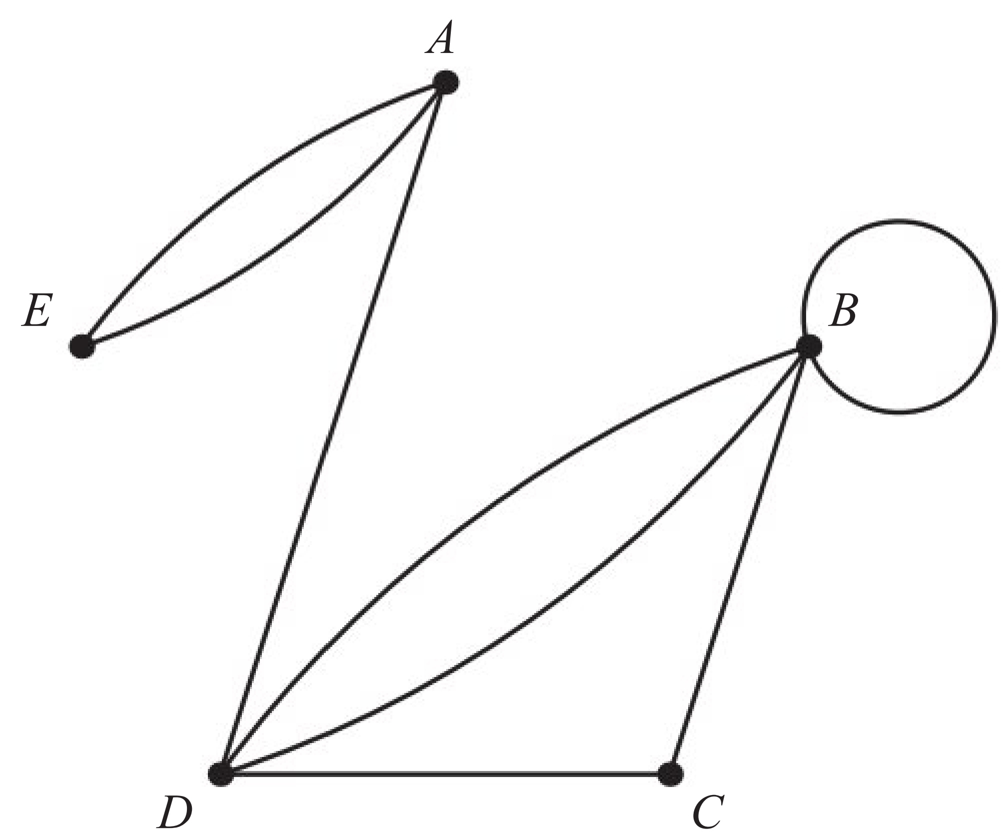
a the out degree of D

b the in degree of A.

73 State which of the graphs below are trees.



74 Write down an adjacency matrix for the graph shown below.



75 Draw a directed graph represented by the following adjacency matrix. The rows represent the 'from' vertices and the columns represent the 'to' vertices.

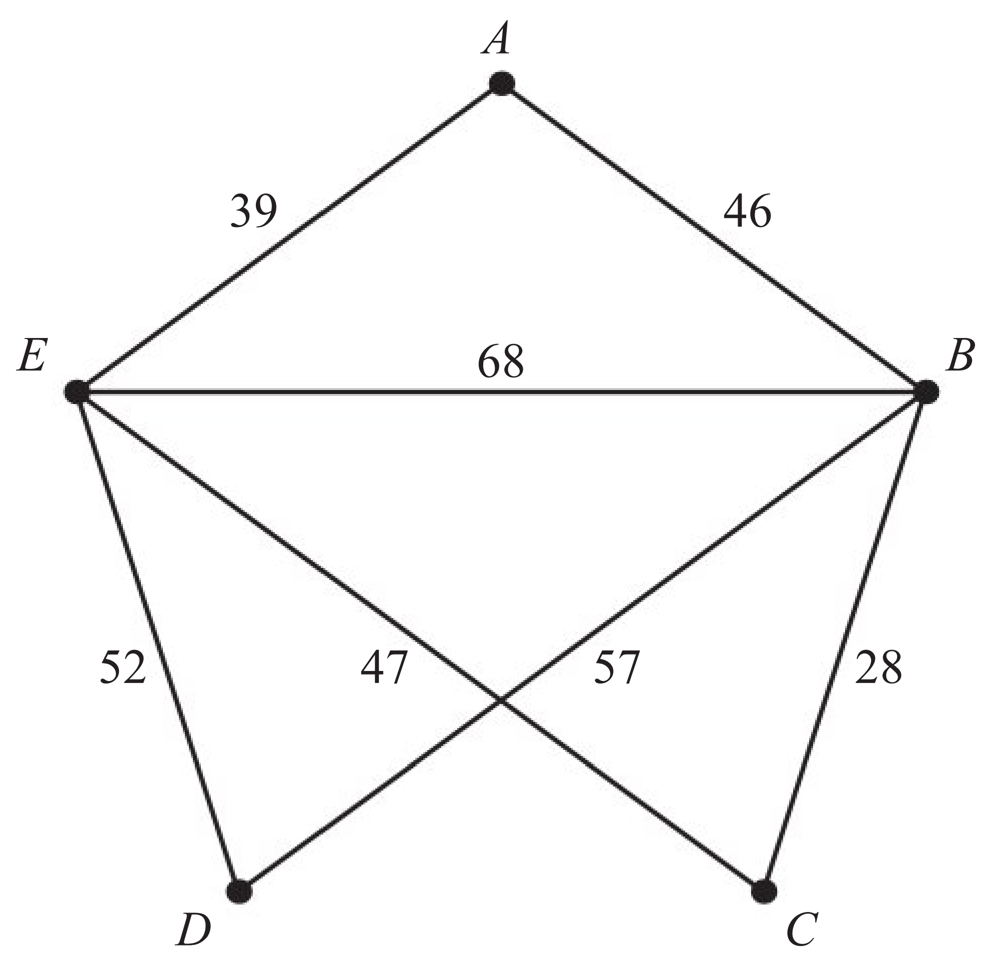
	A	**B**	**C**	**D**
A	0	1	0	1
B	1	0	1	0
C	1	0	0	1
D	1	1	0	0

76 For the graph from question **75**, find:

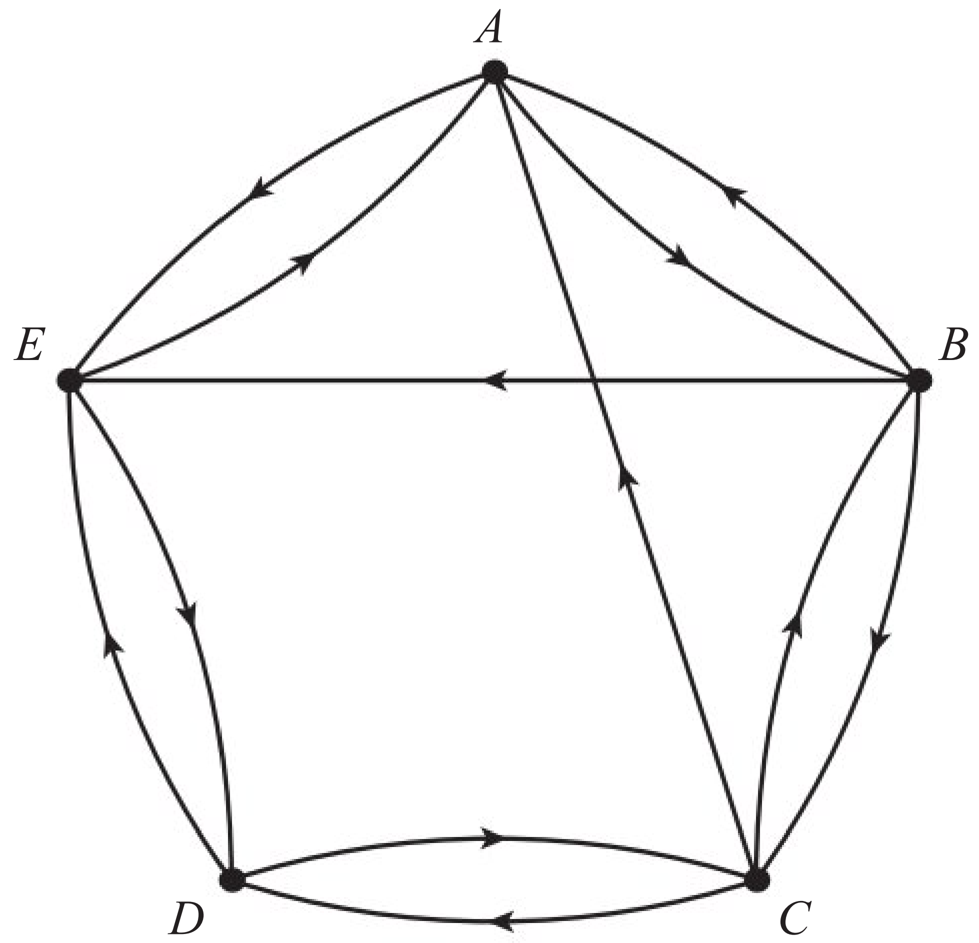
a the number of walks of length 10 starting at A and finishing at C

b the number of walks of length 5 starting and finishing at D.

77 Construct a weighted adjacency table for the graph below.



78 a Construct a transition matrix for the directed graph below.

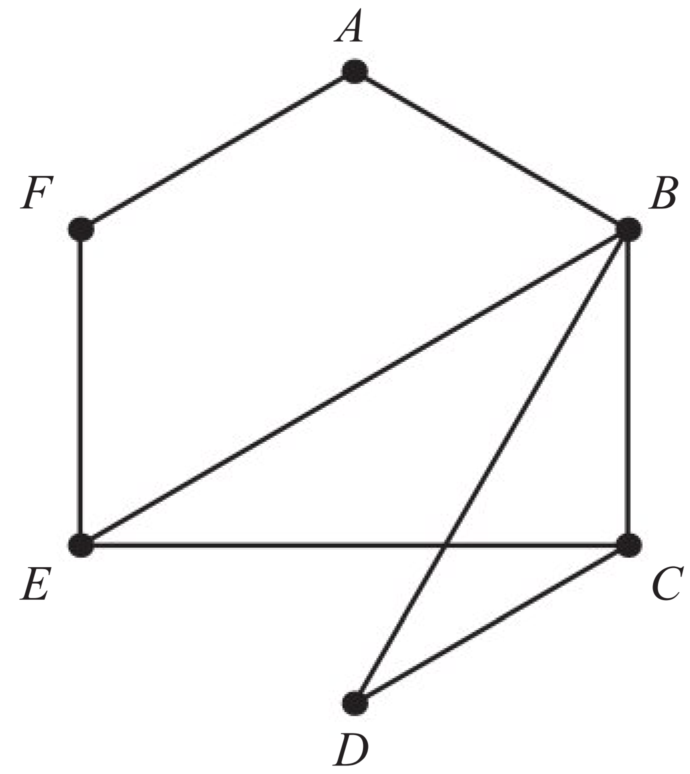


b A random walk starts at vertex B. What is the probability that it is at B again after 8 steps?

79 The graph in question **78** represents links between web pages. Use the PageRank algorithm to determine which page should be ranked top.

80 Fill in the table for the walks shown in the graph on the right, putting 'Yes' or 'No' in each cell:

	Trail	Path	Circuit	Cycle
BCDBE				
BAFEB				
CDBAFEC				
CDBCEBAFE				
BDCEBC				

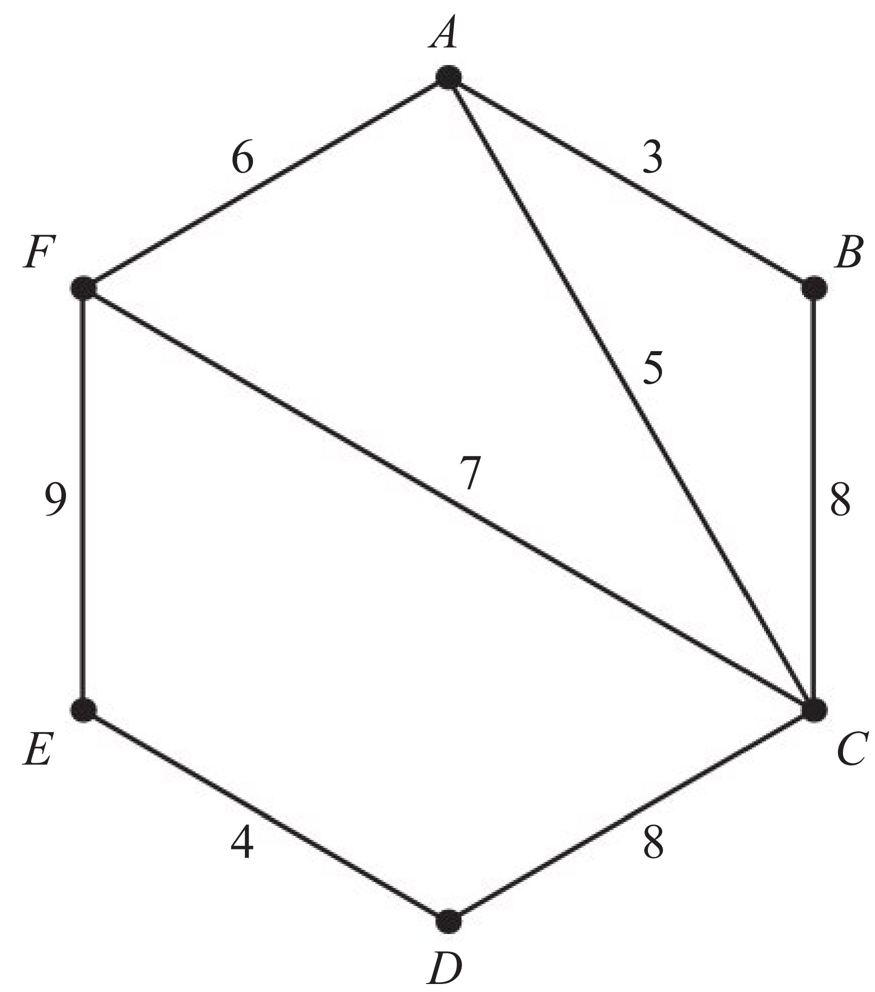


81 **a** Explain, giving reasons, whether the graph in question **80** has an Eulerian cycle.

b Find an Eulerian trail, explaining your choice of the start and end vertices.

82 For the graph in question **80**, write down all the Hamiltonian cycles starting and finishing at A.

83 Use Kruskal's algorithm to find the minimum spanning tree for the graph below.
List the edges in the order you add them.
State the weight of your tree.

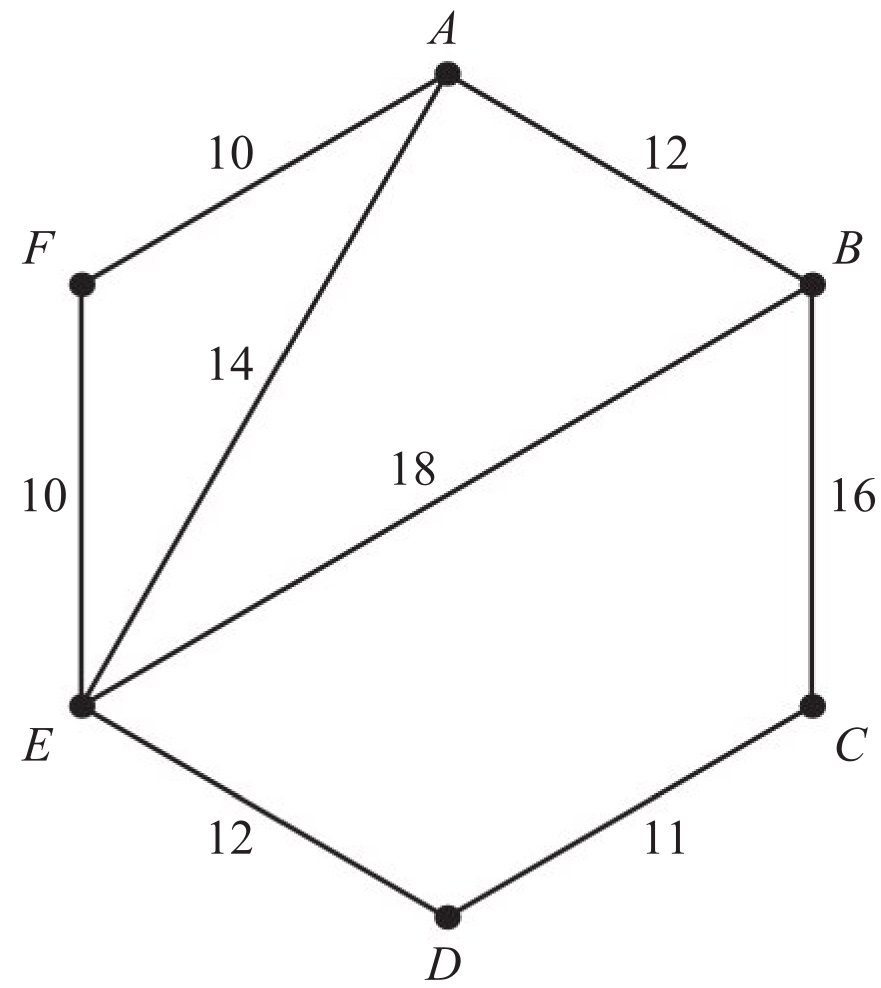


84 Use Prim's algorithm, starting from vertex F, to find the minimum spanning tree for the graph in question **83**.
List the edges in the order you add them.
State the weight of your tree.

85 The matrix below represents an undirected graph. Use the matrix form of Prim's algorithm, starting at A, to find the minimum spanning tree for the graph.

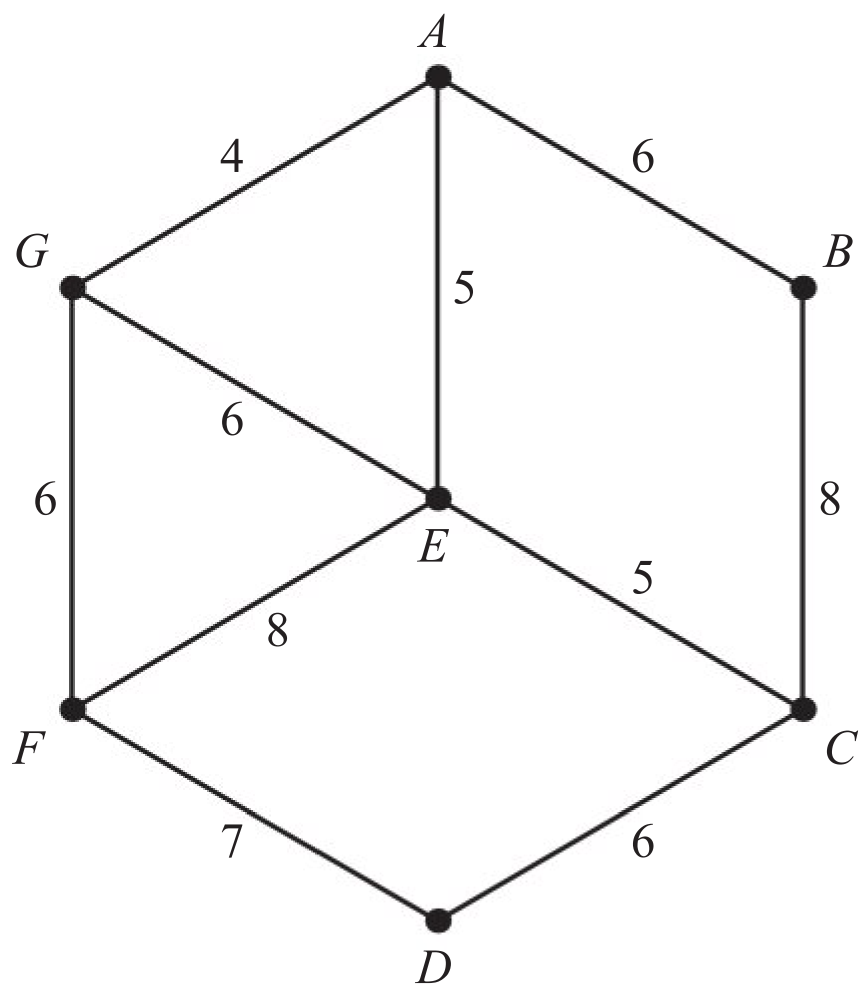
	A	**B**	**C**	**D**	**E**	**F**	**G**
A	–	5	7	–	4	–	8
B	5	–	6	5	–	–	9
C	7	6	–	–	5	3	–
D	–	5	–	–	8	6	2
E	4	–	5	8	–	4	–
F	–	–	3	6	4	–	8
G	8	9	–	2	–	8	–

86 **a** Find the length of the shortest route around the graph below which uses each edge at least once and returns to the starting point.

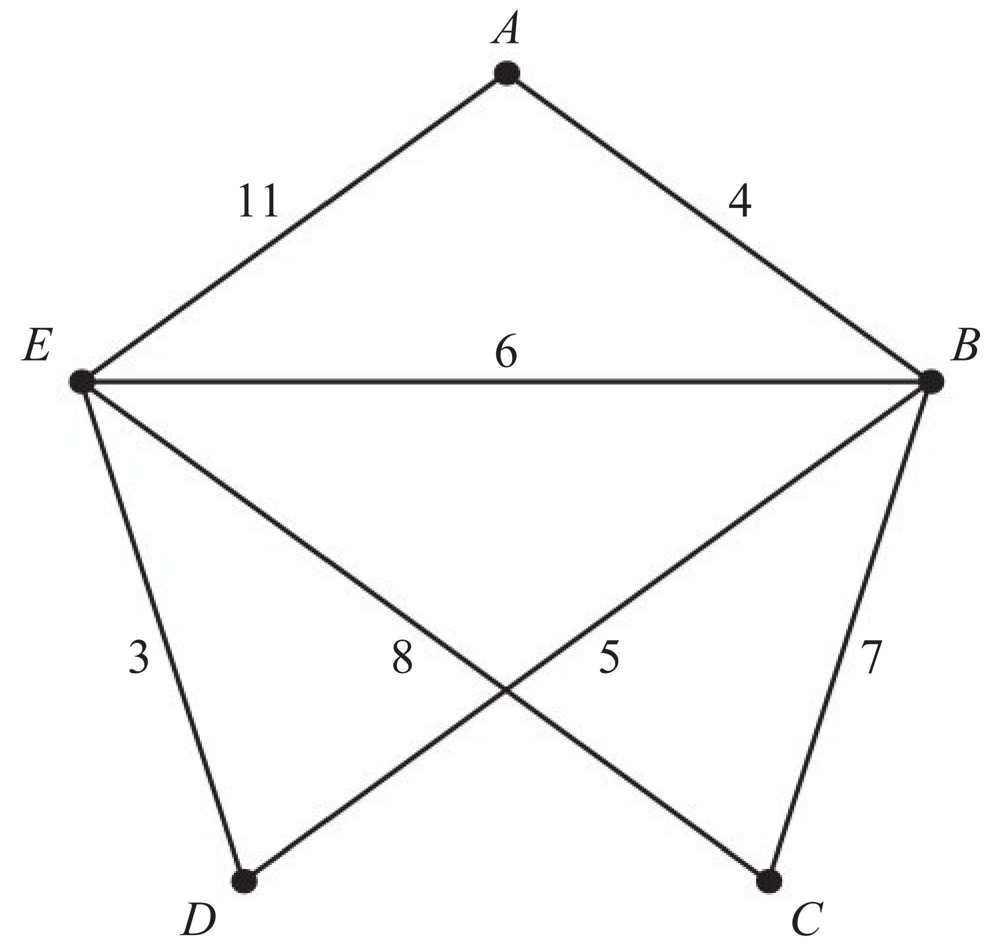


b Another route needs to use each edge exactly once, but it does not need to start and finish at the same point. At which vertices could this route start?

87 Find the length of the shortest route around the graph below which uses each edge at least once and returns to the starting point.



88 Complete a table of least distances for the graph below.



89 Use the nearest neighbour algorithm starting at vertex A to find an upper bound for the travelling salesman problem for the graph from question **88**.

90 By removing vertex D, find a lower bound for the travelling salesman problem for the graph from question **88**.

91 The table shows various upper and lower bounds for the travelling salesman problem for the graph in question **88**, obtained by using different starting vertices.

Starting / deleted vertex	**A**	**B**	**C**	**D**	**E**
Upper bound	31	31	31	35	31
Lower bound	28	29	27	25	25

Write down an inequality for the length L of the Hamiltonian cycle of least weight.

4 Statistics and probability

Syllabus content

S4.1	**Sampling**		
Book Section 6A	Revised ☐		
Syllabus wording	*You need to be able to:*	*Question*	
Concepts of population, sample, random sample, discrete and continuous data.	Identify if data are continuous or discrete.	1	☐
	Identify in context what the population is, what the sample is and whether it is random.	2	☐
Reliability of data sources and bias in sampling.	Identify bias in sampling (a tendency for the sample to include more of one type of object).	3	☐
	Identify reliability of data (strictly the consistency of their results and, in a more colloquial sense, how trustworthy they are).	4	☐
	Deal with missing data or errors in the recording of data.	5	☐
Interpretation of outliers.	Know that an outlier is defined as more than 1.5 × IQR from the nearest quartile, and be able to suggest how to determine if an outlier should be removed from the sample.	6	☐
Sampling techniques and their effectiveness.	Be able to identify and evaluate the following sampling techniques: • simple random • convenience • systematic • quota • stratified.	7	☐
	Calculate the number of data items in each category of a stratified sample.	8	☐

S4.2	**Statistical diagrams**		
Book Section 6C	Revised ☐		
Syllabus wording	*You need to be able to:*	*Question*	
Presentation of data: Frequency distributions.	Interpret frequency distribution tables.	9	☐
Histograms.	Interpret frequency histograms.	10	☐
Cumulative frequency graphs.	Interpret cumulative frequency graphs, including finding median, quartiles, percentiles, range and interquartile range. $IQR = Q_3 - Q_1$	11	☐
Box and whisker plots.	Produce box and whisker diagrams.	12	☐
	Interpret box and whisker diagrams, including using them to compare distributions and use their symmetry to determine if a normal distribution is plausible.	13	☐

S4.3	**Summary statistics**		
Book Section 6B	Revised ☐		
Syllabus wording	*You need to be able to:*	*Question*	
Measures of central tendency.	Calculate the mean, median and mode of data.	14	☐
	Use the formula for the mean of data: $\bar{x} = \frac{\sum_{i=1}^{k} f_i x_i}{n}$ where $n = \sum_{i=1}^{k} f_i$	15	☐
Estimation of mean from grouped data.	Use mid-interval values to estimate the mean of grouped data.	16	☐
Modal class.	Find the modal class for grouped data using tables or histograms.	17	☐
Measures of dispersion.	Use technology to calculate interquartile range (IQR), standard deviation and variance.	18	☐
Effect of constant changes on the original data.	Calculate the mean and standard deviation (and other statistics) of the new data set after a constant change.	19	☐
Quartiles of discrete data.	Use technology to obtain quartiles.	20	☐

S4.4	**Correlation and regression**		
	Book Section 6D	Revised	
Syllabus wording	*You need to be able to:*	*Question*	
Linear correlation of bivariate data: Pearson's product moment correlation coefficient, r.	Calculate the correlation coefficient of bivariate data using technology, and interpret the result, including being aware that correlation does not imply causation.	21	
Scatter diagrams.	Estimate the line of best fit by eye, knowing that it should pass through the mean point.	22	
Equation of the regression line of y on x.	Calculate the equation of the regression line using technology.	23	
Use of the equation of the regression line for prediction purposes.	Use the regression line while being aware of the dangers of extrapolation. Be aware of when a y-on-x regression line is appropriate.	24	
Interpret the meaning of the parameters, a and b, in a linear regression.	Put the meaning of the parameters into context.	25	
Piecewise linear models.	Create and use piecewise linear models.	26	

S4.5	**Definitions in probability**		
	Book Section 7A	Revised	
Syllabus wording	*You need to be able to:*	*Question*	
Concept of trial, outcome, equally likely outcomes, relative frequency, sample space and event.	Estimate probability from observed data.	27	
The probability of an event A is $\mathrm{P}(A) = \frac{n(A)}{n(U)}$.	Find theoretical probabilities by listing all possibilities.	28	
The complementary events A and A'.	Link the probability of an event occurring and it not occurring using: $\mathrm{P}(A) + \mathrm{P}(A') = 1$	29	
Expected number of occurrences.	Calculate how many times an outcome will be observed by multiplying the number of trials and the probability.	30	

S4.6	**Probability techniques**		
	Book Section 7B	Revised	
Syllabus wording	*You need to be able to:*	*Question*	
Use of Venn diagrams, tree diagrams, sample space diagrams and tables of outcomes to calculate probabilities.	Use Venn diagrams to organize information and find probabilities.	31	
	Use tree diagrams to organize information and find probabilities. In tree diagrams you multiply along the branches and add between the branches.	32	
	Use sample space diagrams to organize information and find probabilities.	33	
	Use tables of outcomes to organize information and find probabilities.	34	
Combined events.	Work with the notation $A \cap B$ meaning A and B occurring. Work with the notation $A \cup B$ meaning A or B or both occurring. Use: $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$.	35	
Mutually exclusive events.	Know that mutually exclusive means that the two events cannot both occur, so that $\mathrm{P}(A \cap B) = 0$. Therefore $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B)$.	36	
Conditional probability.	Know that $\mathrm{P}(A \mid B)$ means the probability of A given that B has happened. Use Venn diagrams, tree diagrams, sample space diagrams or tables of outcomes to find conditional probabilities.	37	
Independent events.	Know that if two events, A and B, are independent (that is, do not affect each other) then $\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B)$.	38	

S4.7	Discrete random variables		
	Book Section 8A	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*	
Concept of discrete random variables and their distribution.	Create probability distributions from context.	39	☐
	Use the fact that the total probability in a probability distribution equals 1.	40	☐
Expected value (mean) for discrete data.	Use: $E(X) = \sum xP(X = x)$.	41	☐
Applications.	Use probability distributions to answer questions in context.	42	☐
	Know that $E(X) = 0$ indicates a fair game if X represents the gain of a player.	43	☐

S4.8	Binomial distribution		
	Book Section 8B	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*	
Binomial distribution.	Recognize that if a situation has • a fixed number of trials • outcomes that can be classified into two groups, 'successes' and 'failures' • fixed probability of being in each group • independent trials then the number of successes follows a binomial distribution.	44	☐
	Use technology to calculate binomial probabilities.	45	☐
Mean and variance of the binomial distribution.	Use: $E(X) = np$ $\text{Var}(X) = np(1 - p)$ where X is the number of successes when there are n binomial trials each with a probability p of success.	46	☐

S4.9	Normal distribution		
	Book Section 8C	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*	
The normal distribution and curve; properties of the normal distribution.	Recognize that many natural situations are well modelled by a normal distribution. One way to validate this is to use the fact that about 68% of normally distributed data should fall within one standard deviation of the mean, about 95% within two standard deviations and about 99.7% within three standard deviations.	47	☐
Diagrammatic representation.	Recognize that a normal distribution can be represented by a symmetric bell-shaped curve with area representing probability.	48	☐
Normal probability calculations.	For a given mean and standard deviation, find the probability of a random variable falling in a given interval.	49	☐
Inverse normal calculations.	For a given probability, find the boundary of the region it describes.	50	☐

S4.10	Spearman's rank correlation coefficient		
	Book Section 15C	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*	
Spearman's rank correlation coefficient, r_s.	Calculate Spearman's rank using technology.	51	☐
	Average rank of equally ranked items.	52	☐
Awareness of the appropriateness and limitations of Pearson's and Spearman's correlation coefficients.	Choose an appropriate correlation coefficient, justifying your choice (Pearson's when testing for linearity, Spearman's for any monotonic relationship).	53	☐
	Understand that Spearman's is less sensitive to outliers than Pearson's.	54	☐

S4.11	Hypothesis testing		
	Book Section 15A, 15B	Revised ☐	
Syllabus wording		*You need to be able to:*	*Question*
Null and alternative hypotheses, significance levels and p-values.		Write down appropriate hypotheses for a given situation.	55 ☐
Expected and observed frequencies.		Find expected frequencies when a given ratio is expected.	56 ☐
		Find expected frequencies for a binomial distribution.	57 ☐
		Find expected frequencies for a normal distribution.	58 ☐
χ^2 test for independence: contingency tables; degrees of freedom; critical value.		Use technology to find the p-value and the χ^2 statistic, including determining the degrees of freedom.	59 ☐
χ^2 test for goodness of fit.		Use technology to find the p-value and the χ^2 statistic, including determining the number of degrees of freedom.	60 ☐
		Use a given critical value to determine a conclusion.	61 ☐
The t-test.		Use a t-test to determine if a population mean has changed from a prior belief using a given sample.	62 ☐
		Use a t-test to determine if a population mean has changed from a prior belief using summary statistics.	63 ☐
Use of the p-value to compare the means of two populations using one-tailed and two-tailed tests.		Use technology to find the p-value (using pooled two-sample t-test) and interpret the result of the test.	64 ☐
		Distinguish between one-tail and two-tail tests.	65 ☐
		Understand that a t-test is only valid if the underlying distributions are normal.	66 ☐

H4.12	Data collection		
	Book Section 9A, 9B	Revised ☐	
Syllabus wording		*You need to be able to:*	*Question*
Design of valid data collection methods, such as surveys and questionnaires.		Be able to criticize data collection methods in terms of both sampling and question design.	67 ☐
Selecting relevant variables from many variables.		Be able to suggest in context which variables are relevant to a particular research question, including considering how best to scale a variable.	68 ☐
Choosing relevant and appropriate data to analyse.		Be able to scan data for obvious errors or signs that data might not be from the population of interest.	69 ☐
Categorizing numerical data in a χ^2 table and justifying the choice of categorization.		Be able to regroup numerical data to ensure that the expected frequencies are greater than 5.	70 ☐
Choosing an appropriate number of degrees of freedom when estimating parameters from data when carrying out the χ^2 goodness of fit test.		Conduct chi-squared tests when parameters are estimated from the data.	71 ☐
Definition of reliability and validity.		Be able to define reliability and validity.	72 ☐
Reliability tests.		Be able to suggest in context a way of checking the reliability of a study.	73 ☐
Validity tests.		Be able to suggest in context a way of checking the validity of a study.	74 ☐

H4.13 Non-linear regression

Book Section 9C | Revised ☐

Syllabus wording	You need to be able to:	Question
Evaluation of least squares regression curves using technology.	Find quadratic regression curves.	75 ☐
	Find cubic regression curves.	76 ☐
	Find exponential regression curves.	77 ☐
	Find power regression curves.	78 ☐
	Find sine regression curves.	79 ☐
Sum of square residuals (SS_{res}) as a measure of fit for a model.	Interpret SS_{res} as a measure of goodness of fit of a regression line.	80 ☐
The coefficient of determination.	Evaluate R^2 using technology, including using it to compare models.	81 ☐
	Find R^2 for linear models from the correlation coefficient.	82 ☐

H4.14 Combinations of random variables

Book Section 8A | Revised ☐

Syllabus wording	You need to be able to:	Question
Linear transformation of a single random variable.	Find the expected value of a linear transformation of a random variable: $\mathrm{E}(aX + b) = a\,\mathrm{E}(X) + b$.	83 ☐
	Find the variance of a linear transformation of a random variable using: $\mathrm{Var}(aX + b) = a^2\,\mathrm{Var}(X)$.	84 ☐
Expected value of linear combinations of n random variables.	Find the expected value of linear combinations of n random variables using: $\mathrm{E}(a_1X_1 \pm a_2X_2 \pm \cdots \pm a_nX_n) = a_1\,\mathrm{E}(X_1) \pm a_2\,\mathrm{E}(X_2) \pm \cdots \pm a_n\,\mathrm{E}(X_n)$	85 ☐
Variance of linear combinations of n independent random variables.	Find the variance of linear combinations of n random variables using: $\mathrm{Var}(a_1X_1 \pm a_2X_2 \pm \cdots \pm a_nX_n) = a_1^2\,\mathrm{Var}(X_1) + a_2^2\,\mathrm{Var}(X_2) + \cdots + a_n^2\,\mathrm{Var}(X_n)$.	86 ☐
$\bar{x}$ as an unbiased estimate of μ.	Find an unbiased estimate of μ from data.	87 ☐
S_{n-1}^2 as an unbiased estimate of σ^2.	Find an unbiased estimate of σ^2 from data.	88 ☐
	Find an unbiased estimate of S_{n-1}^2 from S_n using: $S_{n-1}^2 = \frac{n}{n-1}S_n^2$.	89 ☐

H4.15 Sampling distribution of $\bar{X}$

Book Section 8B | Revised ☐

Syllabus wording	You need to be able to:	Question
A linear combination of n independent normal random variables is normally distributed, in particular: $X \sim N(\mu, \sigma^2) \Rightarrow \bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$	Use the fact that sums and differences of independent normal variables are also normally distributed.	90 ☐
	Use the fact that the sample mean of independent identically distributed normal variables is normally distributed.	91 ☐
Central limit theorem.	Use the fact that if $n > 30$, the sample mean will approximately follow a normal distribution.	92 ☐

H4.16	Confidence intervals		
	Book Section 9D	Revised ☐	
Syllabus wording		*You need to be able to:*	*Question*
Confidence intervals for the mean of a normal population.		Use technology to find a z-interval when σ is known.	93 ☐
		Use technology to find a t-interval when σ is unknown.	94 ☐
		Interpret confidence intervals in context.	95 ☐

H4.17	Poisson distribution		
	Book Section 8C	Revised ☐	
Syllabus wording		*You need to be able to:*	*Question*
Poisson distribution, its mean and variance.		Recognize situations in which it is appropriate to use a Poisson distribution as a model.	96 ☐
		Know that for a Poisson distribution, the mean equals the variance.	97 ☐
		Use technology to calculate probabilities for a Poisson distribution.	98 ☐
Sum of two independent Poisson distributions has a Poisson distribution.		Use the fact that the sum of two independent Poisson distributions also has a Poisson distribution.	99 ☐

H4.18	Further hypothesis testing		
	Book Section 9E, 9F, 9G	Revised ☐	
Syllabus wording		*You need to be able to:*	*Question*
Critical values and critical regions.		Understand the terminology critical region and critical value.	100 ☐
Test for population mean for normal distribution.		Use a z-test when the population variance is known.	101 ☐
		Conduct paired t-tests or z-tests.	102 ☐
		Find critical regions for z-tests.	103 ☐
Test for proportion using binomial distribution.		Conduct a one-tailed test to see if a sample could plausibly be drawn from a population with a given proportion of successes.	104 ☐
		Find the critical region for binomial tests.	105 ☐
Test for population mean using Poisson distribution.		Conduct a one-tailed test to see if a sample could plausibly be drawn from a population with a given mean.	106 ☐
		Find the critical region for Poisson tests.	107 ☐
Use of technology to test the hypothesis that the population product moment correlation coefficient (ρ) is 0 for bivariate normal distributions.		Use GDC to conduct a test on the sample correlation coefficient.	108 ☐
Type I and II errors including calculations of their probabilities.		Understand the meaning of type I and type II errors in context.	109 ☐
		Know the relationship between significance level and probability of a type I error for a continuous random variable.	110 ☐
		Calculate the probability of a type II error for a Z-test, given the true value of the mean.	111 ☐
		Calculate the probability of a type I error for a binomial or Poisson test.	112 ☐
		Calculate the probability of a type II error for a binomial or Poisson test.	113 ☐

H4.19	Markov chains		
	Book Section 8D	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*	
Transition matrices.	Use descriptions or transition diagrams to write down transition matrices.	114	☐
	Use transition matrices to draw transition diagrams.	115	☐
Powers of transition matrices.	Use technology to work out the state after a given number of iterations.	116	☐
Regular Markov chains. Initial state probability matrices.	Understand the terms 'Regular Markov chain' and 'Initial state probability matrices'.	117	☐
Calculation of steady state and long-term probabilities by repeated multiplication of the transition matrix or by solving a system of linear equations.	Find the long term state of a Markov chain process.	118	☐

Practice questions

1 Determine whether each of the following variables is continuous or discrete:

a Number of people in a family.

b Time for a nucleus to decay.

c Age in complete years.

2 A doctor wants to find out whether exercise can lower the incidence of illness. He asks patients who come to his clinic to fill in a survey about their exercise habits; 20% of them agree to do this.

a Suggest a possible population that the doctor is interested in.

b Is his sample random?

3 Is the sampling in question **2** likely to be biased? Justify your answer.

4 Five independent groups of people were asked to estimate the length of an arrow which is 5 cm long. The averages for the groups were 4.6 cm, 4.6 cm, 4.7 cm, 4.8 cm, 4.8 cm. Does this suggest that the results are reliable?

5 Five people were asked to record their height in metres:
A: 1.83 B: 1.45 C: 1.77 D: 5.10 E: 1.60
Suggest which data item is an error. What should be done with this item?

6 A data set has lower quartile 7 and upper quartile 11. Explain why 18 should be considered an outlier and suggest how to determine if it should be excluded from the data.

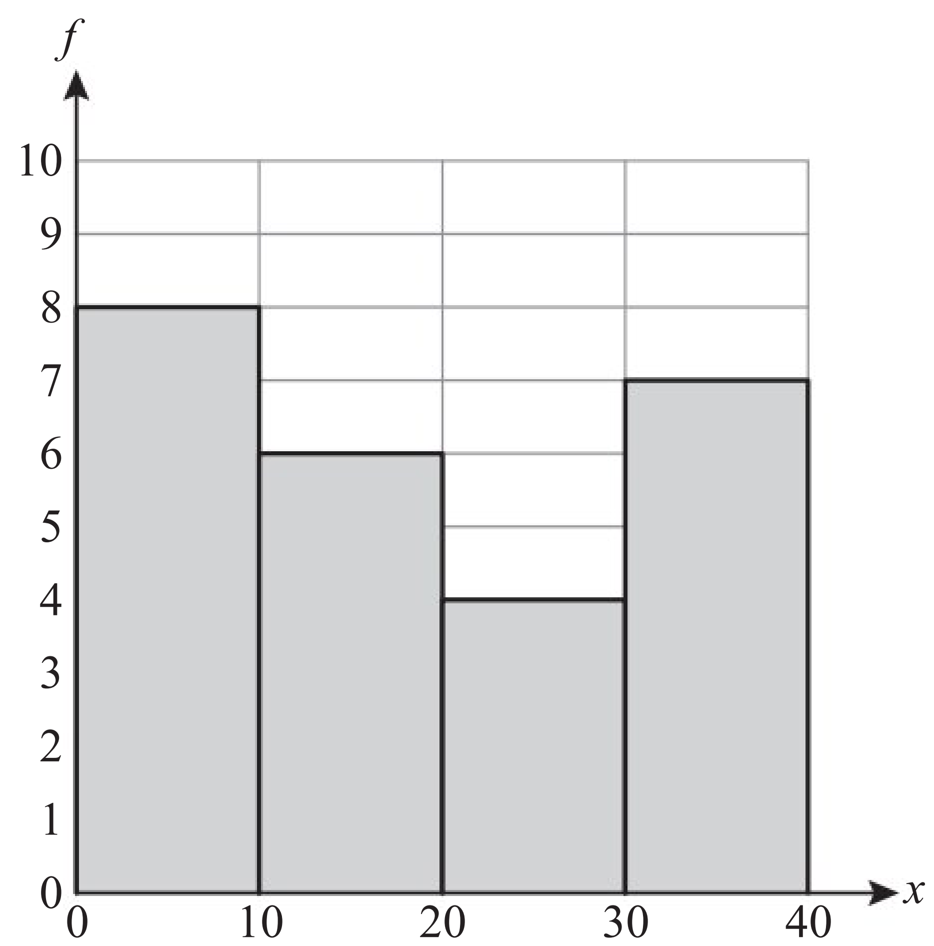
7 Write down the sampling method used by the doctor in question **2**.

8 A language school consists of students from either Italy or Spain. There are 60 from Italy and 90 from Spain. In a stratified sample of 20 students, how many should be from Italy?

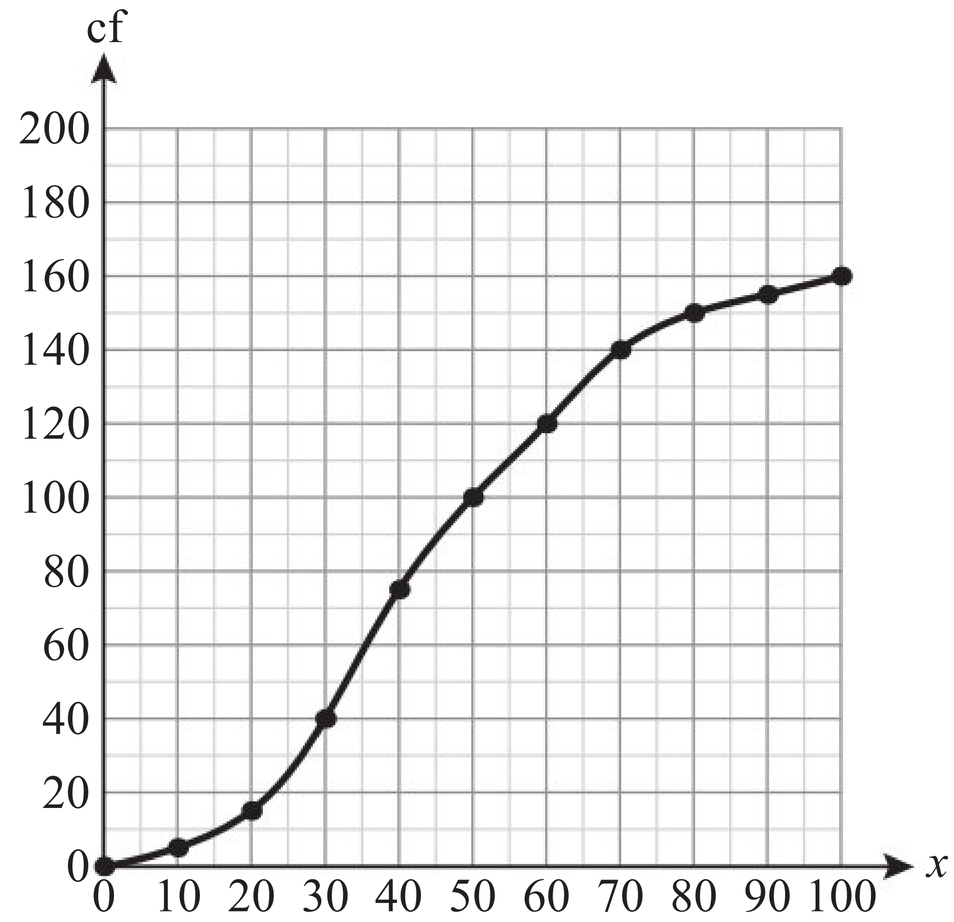
9 For the following frequency table, find the proportion of data items above 20.

x	$0 < x \le 20$	$20 < x \le 30$	$30 < x \le 40$
Frequency	15	18	12

10 For the following histogram, estimate the number of data items above 25.



11 The following diagram shows a cumulative frequency graph.



Find:

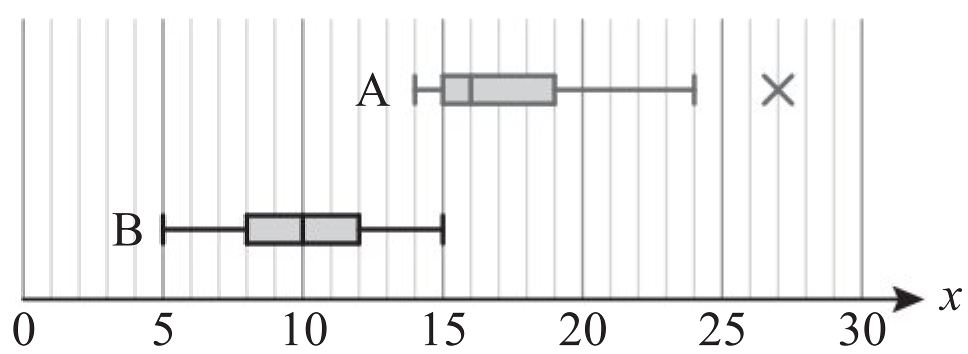
a the median

b the interquartile range

c the 90th percentile.

12 Sketch a box and whisker plot for the sample below:
12, 13, 15, 16, 16, 18, 18, 19, 20.

13 The following diagram shows two box and whisker plots.



a Compare the two distributions.

b Determine, with justification, which of the two distributions is more likely to be a normal distribution.

14 Write down the mean, median and mode of the following data:
14, 14, 16, 17, 19, 20, 23, 25

15 The numbers 4, 8, 2, 9 and x have a mean of 7. Find the value of x.

16 **a** Estimate the mean of the following grouped data.

x	$10 < x \leq 20$	$20 < x \leq 30$	$30 < x \leq 50$	$50 < x \leq 60$
Frequency	10	12	15	13

b Explain why it is only an estimate.

17 Find the modal class for the data below:

x	$0 < x \leq 5$	$5 < x \leq 10$	$10 < x \leq 15$	$15 < x \leq 20$
Frequency	16	12	15	18

18 For the data set 6, 7, 9, 12, 14, 18, 22, find:

a the interquartile range

b the standard deviation

c the variance.

19 A set of data has mean 12 and standard deviation 10. Every item in the data set is doubled, then 4 is added on. Find the mean and standard deviation of the new data set.

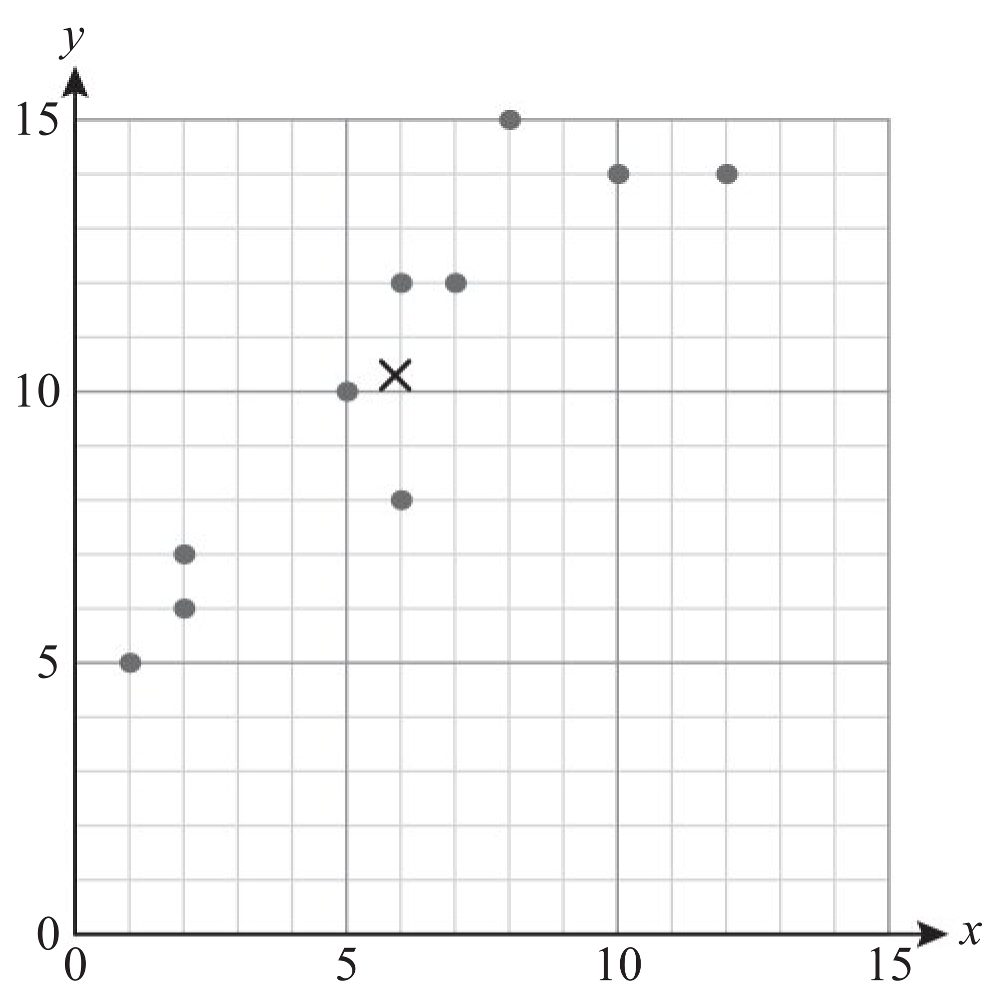
20 Find the quartiles of the following data:
17, 15, 23, 29, 15, 22, 28, 30

21 **a** Calculate Pearson's product moment correlation coefficient for the following data.

x	2	4	4	7
y	1	3	6	8

b Interpret your result.

22 The following diagram shows a set of 10 data items with the mean point labelled with a cross.



a Sketch a line of best fit on the diagram.
b Hence estimate y when $x = 3$.

23 Find the y-on-x regression line for the following data, in which y is dependent on x.

x	1	2	2	5	6	6	7	8	10	12
y	5	6	7	10	12	8	12	15	14	14

24 **a** Use your answer to question **23** to estimate:

i y when $x = 9$

ii y when $x = 20$

iii x when $y = 10$.

b Which of the predictions made in part **a** is valid? Justify your answer.

25 A social scientist investigates how the number of text messages sent by pupils each day (y) depends on the number of hours they spend on social media each day (x). He finds the regression line:

$$y = 6.7 + 1.4x$$

Interpret what each of the following numbers means in context.

a 6.7

b 1.4

26 A veterinary researcher believes that the growth of a breed of snake is very different during their first 6 months compared to their next 6 months.

She collects the following data showing the length (L cm) and age (A months) of a sample of snakes.

A	1	2	4	4	7	7	10	11	12
L	4	8	15	18	30	32	34	36	34

a Create a piecewise linear model to reflect the researcher's belief.

b Use your answer to part **a** to estimate the length of a 3-month-old snake of this breed.

27 A coin is flipped 200 times and 134 heads are observed. Estimate the probability of observing a head when the coin is flipped.

28 Find the probability of rolling a prime number on a fair six-sided dice.

29 If $P(A) = 0.6$, find $P(A')$.

30 If there are 30 pupils in a class and the probability of a student being absent is 0.05, find the expected number of absent pupils.

31 In a class of 30 students, 14 study French, 18 study Spanish and 4 study both languages. Find the probability that a randomly chosen student studies neither French nor Spanish.

32 A drawer contains three white socks and five black socks. Two socks are drawn without replacement.

a Find P(2nd sock is black|1st sock is white).

b Find the probability that the socks are different colours.

33 A fair four-sided dice is thrown twice.

a What is the probability that the total score is greater than 5?

b If the total score is greater than 5, what is the probability that it is 7?

34 One hundred students were asked whether they preferred soccer or cricket. They were also asked if they preferred maths or art. The results are summarized in the table:

	Soccer	**Cricket**
Maths	40	20
Art	30	x

a Find the value of x.

b Find the probability that a randomly chosen student prefers maths to art.

35 If $P(A) = 0.5$, $P(B) = 0.7$ and $P(A \cap B) = 0.3$, find $P(A \cup B)$.

36 Events A and B are mutually exclusive. If $P(A) = 0.4$ and $P(B) = 0.2$, find $P(A \cup B)$.

37 For the sample in question **34**, determine the probability that a randomly chosen person who prefers soccer also prefers maths.

38 Independent events A and B are such that $P(A) = 0.4$ and $P(B) = 0.6$. Find $P(A \cap B)$.

39 A drawer contains three white socks and four black socks. Two socks are drawn at random without replacement. Find the probability distribution of W, the number of white socks drawn.

40 The random variable X can take values 0, 1 or 2 with probability $P(X = x) = k(x + 1)$. Find the value of k.

41 For the distribution given below, find E(X).

x	0.5	1	2.5
P($X = x$)	0.5	0.4	0.1

42 The value of prizes (\$$X$) won by an individual each month in a prize draw is shown in the table.

X	0	10	2000
P($X = x$)	0.9	0.095	0.005

a Given that an individual wins a prize, find the probability that it is \$2000.

b Find the probability of winning more than the expected amount.

43 The gain, \$$X$, of a player in a game of chance follows the distribution shown below.

X	–1	0	k
P($X = x$)	0.6	0.3	0.1

Find the value of k that would make the game fair.

44 A drawer contains 5 black socks and 10 red socks; 4 socks are drawn at random without replacement. Explain why the number of black socks drawn does not follow a binomial distribution.

45 If X is a random variable following a binomial distribution with five trials and a probability of success of 0.4, find:

a P($X = 2$)

b P($X \geq 3$).

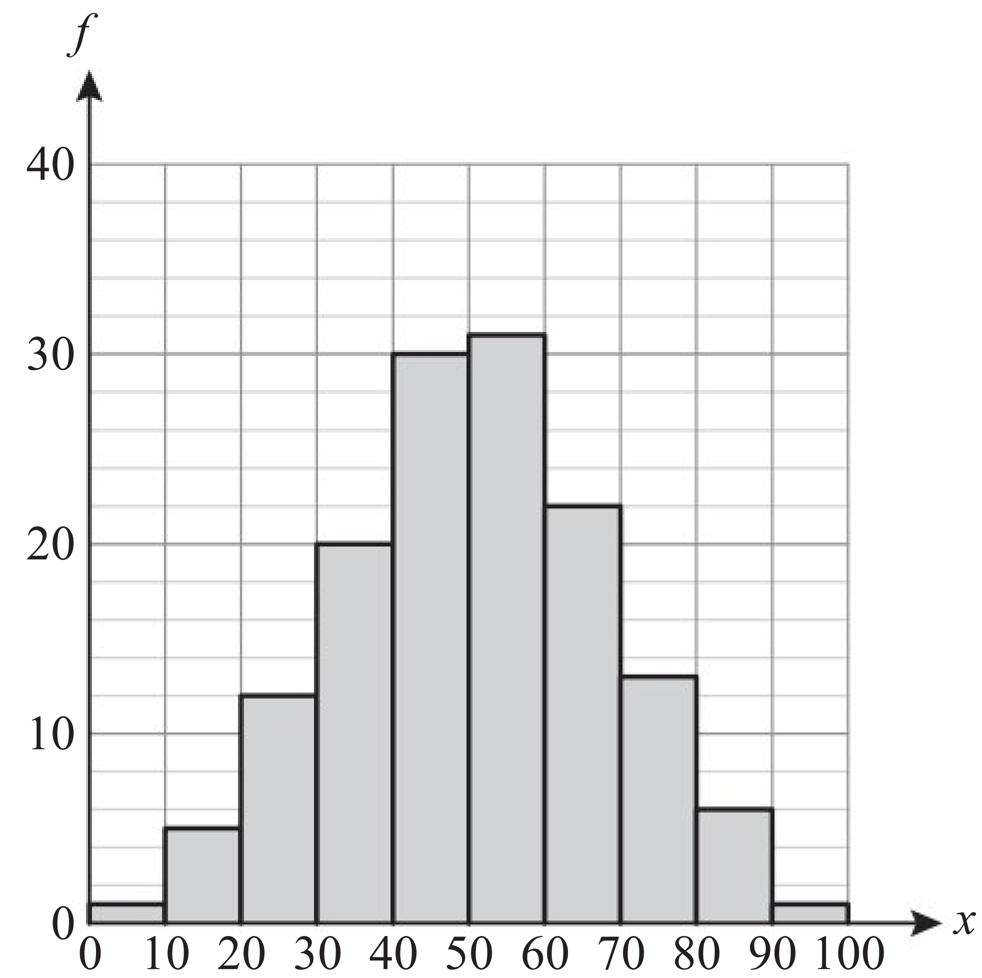
46 A biased coin has a probability of 0.6 of showing a head. It is flipped 10 times.
If this experiment is repeated many times, find:

a the expected mean number of heads

b the expected standard deviation in the number of heads.

47 The time for a child to learn a new dance move is found to have a mean of 2 weeks and a standard deviation of 4 weeks. Explain why this variable is unlikely to be modelled by a normal distribution.

48 The following histogram shows the results of an experiment.



a What feature of this graph suggests a normal distribution might be a good model for the outcome of the experiment?

b Visually estimate the mean of the distribution.

49 A random normal variable has mean 12 and standard deviation 2. Find the probability that an observation is between 11 and 15.

50 A random normal variable has mean 100 and standard deviation 15. The probability of being above k is 0.7. Find the value of k.

51 Calculate the Spearman's rank correlation coefficient for the following data set.

x	1	4	3	8
y	2	–1	0	3

52 Calculate the Spearman's rank correlation coefficient for the following data set.

x	0	5	5	7
y	4	4	7	4

53 A scientist wants to test to see if as x increases, y also tends to increase. Would r or r_s be a more appropriate correlation statistic to use?

54 A scientist is concerned that her data include outliers. Should she use r or r_s to test for possible correlation?

55 James believes that the mean of a population is 0.4. Write down appropriate null and alternative hypotheses if he wants to determine if the mean has:

a changed

b increased.

56 A geneticist believes that the ratio of blond to brown to black hair should be 1:4:3. He collects a sample of size 100. Find the expected values of each hair colour.

57 Thirty-five experiments are conducted. The outcome of each experiment is a number which is believed to be taken from a binomial distribution with two trials and a probability 0.4 of success. Find the expected frequencies of the experimental outcomes.

58 The following data were observed:

x	$100 \le x < 120$	$120 \le x < 130$	$130 \le x < 140$	$140 \le x < 150$
Frequency	30	35	20	5

Find the expected frequencies if the data are drawn from a normal distribution with mean 125 and standard deviation 10.

59 The following data were collected for variables X and Y:

		Variable X		
		A	B	C
Variable Y	D	12	24	16
	E	32	15	18

a Assuming that variables X and Y are independent, find the expected values.

b Conduct a hypothesis test at the 5% significance level to see if the two variables are dependent.

60 The following table shows the observed frequencies from an experiment.

Outcome	A	B	C	D
Observed frequency	10	15	12	13

Is there evidence, at the 5% significance level, that the outcomes are not all equally likely?

61 The outcome of a chi-squared goodness of fit test is 15.3. A table of critical values says that the appropriate critical value is 14.07. Is there significant evidence that the observed frequencies differ from the expected frequencies? Justify your answer.

62 Does the following sample suggest, at the 5% significance level, that the population mean is bigger than 10?
14, 9, 12, 11, 15

63 Kwami wants to test if the mean length of newts is different from the 13 cm he read in a textbook. Based on a sample of 12 newts he finds that $\bar{x} = 11.8$ and $s_{n-1} = 1.2$. Determine if there is evidence, at the 5% significance level, of a difference from the textbook value.

64 Determine whether there is evidence at the 10% significance level that the two groups below are drawn from populations with different means.

Group A	13	18	19	16
Group B	8	15	20	20

65 For the following data determine, at the 5% significance level, if group A is drawn from a population with a larger mean than group B.

Group A	23	16	19
Group B	11	16	14

66 State one distributional assumption required when using a t-test.

67 Giulio works for a town council. He decides to use an internet survey to decide whether to close a nursery. One of his questions is:
'Do you agree that budgets must be balanced and childcare provision should focus on those parents who need it most? YES/NO (Delete as appropriate)'
Make a criticism of this survey based on:

a the validity of the sampling method

b the validity of the question wording.

68 Samir wants to compare the health provision of different countries. Explain why comparing the number of doctors in the different countries is not a valid comparison. Suggest an improved measure which Samir could use.

69 The following shows an extract from a survey of high school students in 2021. One of the questions shows dates of birth in the format DD/MM/YY.

ID	Date of birth
A	27/11/07
B	01/07/79
C	12/29/08
D	12/04/06
E	22/10/08

Two of these entries contain errors. Suggest, with reasons, which two entries contain errors.

70 Regroup the following data to make them suitable for a χ^2 analysis.

Values	$0 \le x < 5$	$5 \le x < 10$	$10 \le x < 15$	$15 \le x < 20$
Observed frequency	4	7	8	6
Expected frequency	6.1	7.5	6.6	4.8

71 The following data are thought to be drawn from a binomial distribution with four trials. Conduct an appropriate chi-squared test at the 5% significance level, stating the degrees of freedom of the test.

Successes	0	1	2	3	4
Observed frequency	31	48	54	46	21

72 **a** Define reliability.

b Define validity.

73 A teacher wants to test to see how much statistics his students understand. He sets them a 20 question multiple choice test and most students score less than 10 out of 20.
How can the teacher check to see if his conclusion that they understand less than half of the material is reliable?

74 A psychologist wants to measure students' motivation. She uses a questionnaire which contains multiple choice questions to infer their motivation.
How could she test the validity of this questionnaire?

75 For the following data, find a quadratic regression curve in the form $y = ax^2 + bx + c$.

x	y
0	0.1
0	0.2
1	–1.2
2	–2.4
3	–1.4
4	–0.2
4	0.1
5	1.3

76 For the following data, find a cubic regression curve in the form $y = ax^3 + bx^2 + cx + d$.

x	y
0	4.8
1	6.9
2	20.4
4	140.2
4	149.2
6	458.8

77 For the following data, find an exponential regression curve of the form $y = ae^{bx}$.

x	y
–2	0.74
–1	1.18
–1	1.24
0	1.97
0	2.04
1	3.24
2	5.34
3	9.01

78 For the following data, find a power regression curve of the form $y = ax^b$.

x	y
1	1.8
1	2
2	31
3	156
3	166
4	503

79 For the following data, find a sine regression curve in the form $y = a \sin(bx + c) + d$.

x	y
0	3.4
1	4.86
2	4.61
3	2.88
4	1.26
5	1.23

80 An exponential regression curve of the form $y = ae^{bx}$ has $SS_{res} = 132.2$. A power regression curve of the form $y = ax^b$ has $SS_{res} = 105.6$ for the same data set. Based on this, which regression curve is a better fit to the data? Why would it not be appropriate to use SS_{res} to compare the fit of two different data sets?

81 Use R^2, the coefficient of determination, to compare the models formed in questions **77** and **78**. Explain why it would not be appropriate to use R^2 to compare the power regression curve from question **78** with the sine regression curve from question **79**.

82 A linear model has a Pearson product moment correlation coefficient of –0.72. An exponential regression model of the form $y = ae^{bx}$ has a coefficient of determination of 0.62. Find, with justification, which model has a better fit.

83 If $E(X) = 2$, find $E(3X - 5)$.

84 If $Var(X) = 7$ find $Var(1 - 2X)$.

85 If $E(X) = 3$ and $E(Y) = -2$ find $E(2X - Y + 1)$.

86 If $Var(X) = 3$ find $Var\left(\frac{x_1 + x_2}{2}\right)$ where X_1 and X_2 are two independent observations of X.

87 Five measurements of X are shown:
3, 5, 7, 8, 10
Find an unbiased estimate for the population mean of X.

88 For the data in question **87**, find an unbiased estimate for the population variance of X.

89 The standard deviation, S_n, of a sample of size 4 is 12. Find an unbiased estimate for the population variance.

90 The mass of apples is normally distributed with mean 100 g and standard deviation 10 g. The mass of pears is normally distributed with mean weight 160 g and standard deviation 20 g.
What is the probability that a randomly chosen pear is more than twice as heavy as a randomly chosen apple?

91 X is a normally distributed random variable with mean 40 and standard deviation 5.
Find the probability that the mean of 10 observations of X is below 38.

92 The random variable X has mean 12 and variance 20.
Find the probability that the sample mean of 50 independent observations of X is above 11.

93 A sample of size 5 has mean 12.2. It is drawn from a population which is known to have a variance of 16.
Find a 95% confidence interval for the population mean.

94 For the following sample, find a 90% confidence interval for the population mean.
12, 14, 18, 23, 23, 29

95 A paper reports that the 95% confidence interval for the change in blood pressure of people in a clinical trial is $-23 < \mu < 2$.
Does this suggest a decrease in the blood pressure?

96 Explain why the number of worms per square metre in a forest is unlikely to follow a Poisson distribution.

97 X is a random variable that follows a Poisson distribution with mean 2.5. By considering the mean and variance of $2X$, explain why $2X$ does not follow a Poisson distribution.

98 If X follows a Poisson distribution with mean 3.1, find $P(X \leq 3)$.

99 The number of bees observed each hour in a garden follows a Poisson distribution with mean 2.1. The number of butterflies observed each hour in the same garden follows a Poisson distribution with mean 4.3.
Assuming these two variables are independent, and that bees and butterflies are the only insects observed, find the probability that more than 8 insects are observed.

100 For a test with test statistic X, the critical region is $12 < X < 18$.
If $X = 14$, should the null hypothesis be rejected?

101 In a sample of size 10, the mean is 18. If the population standard deviation is known to be 2, determine if there is significant evidence at the 5% significance level that the population mean is below 20.

102 The table below shows the results for four students in two different assessments.

Student	Andi	Beth	Claude	Deepti
Assessment 1	58	90	88	74
Assessment 2	62	95	86	75

Assuming that the differences in results are normally distributed, perform an appropriate test at 5% significance to see if the results of the assessments are different.

103 A sample of size 16 is used for a 5% significance two-tailed z-test.
Find the critical region for the mean if the null hypothesis is $\mu = 100$ and $\sigma = 20$.

104 It is thought that $\frac{1}{6}$ of rolls of a dice should result in a 1; however, Jane suspects that her dice throws a 1 more often than this. To test this, she rolls the dice 100 times and records 21 1s.
Does this provide evidence at the 5% significance level that there are more 1s than expected?

105 For the situation in question **104**, find the critical region.

106 The number of lions in a nature reserve follows a Poisson distribution. A previous study found that there were 4.1 lions per km^2. An ecologist surveys a 5 km^2 area and observes 14 lions.
Is this evidence, at the 5% significance level, that there are fewer lions per km^2 than when the previous study was conducted?

107 For the situation in question **106**, find the critical region.

108 For the following data, test whether $\rho = 0$ against the alternative hypothesis $\rho \neq 0$. Use a 10% significance level.

x	y
12	14
18	10
20	4
24	6

109 In many judicial systems, it is assumed that people are innocent until proven guilty beyond reasonable doubt. In this context, explain:

a what the null hypothesis is

b what a type I error is

c what a type II error is.

110 A value X is thought to follow a normal distribution with standard deviation 40. A sample of size 25 is taken and the critical region is $\bar{X} < 66.84$ or $\bar{X} > 93.16$

a Find the null and alternative hypotheses.

b Find the significance level of this test.

111 Given that, for the test from question **110**, the true value of μ is 90, find the probability of a type II error.

112 A binomial test is used to determine if a coin is biased towards heads. The null hypothesis is $p = 0.5$, where p is the probability of a head. X is the number of heads observed when the coin is flipped 10 times and the null hypothesis is rejected if $X \geq 9$.
Find the probability of a type I error.

113 Find the probability of a type II error for the test described in question **112** if the true value of p is 0.6.

114 Each day can be described as either wet or dry. If it is wet one day, the probability of it being wet the next day is 0.7. If it is dry one day, the probability of it being dry the next day is 0.6. The probability of it being wet on day n is W_n. The probability of it being dry on day n is D_n. The process can be modelled by

$$\begin{pmatrix} W_{n+1} \\ D_{n+1} \end{pmatrix} = \boldsymbol{T} \begin{pmatrix} W_n \\ D_n \end{pmatrix}$$

Write down the matrix $\boldsymbol{T}$.

115 For a process modelled by

$$\begin{pmatrix} A_{n+1} \\ B_{n+1} \end{pmatrix} = \begin{pmatrix} 0.1 & 0.4 \\ 0.9 & 0.6 \end{pmatrix} \begin{pmatrix} A_n \\ B_n \end{pmatrix}$$

draw an appropriate transition diagram.

116 The initial state of a system is described by the vector $\begin{pmatrix} 0.8 \\ 0.2 \end{pmatrix}$. The transition matrix is $\begin{pmatrix} 0.5 & 0.75 \\ 0.5 & 0.25 \end{pmatrix}$.
Find the state of the system after four iterations.

117 Determine which of the following transition matrices will create a regular Markov chain:

$$\boldsymbol{A} = \begin{pmatrix} 0.4 & 0.6 \\ 0.2 & 0.8 \end{pmatrix} \qquad \boldsymbol{B} = \begin{pmatrix} 0.5 & 0.5 & 0.2 \\ 0 & 0.25 & 0.2 \\ 0.5 & 0.25 & 0.6 \end{pmatrix} \qquad \boldsymbol{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0.5 \\ 0 & 0.5 & 0.5 \end{pmatrix}$$

118 Each day a flower can be open or closed. On day zero it is open. The probability of the flower being open on day n is O_n and the probability of it being closed is C_n. The system is modelled by the Markov chain:

$$\begin{pmatrix} O_{n+1} \\ C_{n+1} \end{pmatrix} = \begin{pmatrix} 0.6 & 0.9 \\ 0.4 & 0.1 \end{pmatrix} \begin{pmatrix} O_n \\ C_n \end{pmatrix}.$$

Find the long-term probability that the flower is open.

5 Calculus

Syllabus content

S5.1	The concepts of a limit and derivative		
	Book Section 9A	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*	
Introduction to the concept of a limit.	Estimate the value of a limit from a table.	1	☐
	Estimate the value of a limit from a graph.	2	☐
Derivative interpreted as gradient function and as rate of change.	Understand and use the notation for derivatives: $\frac{dy}{dx}$ and $f'(x)$.	3	☐
	Interpret the derivative as a rate of change.	4	☐
	Interpret the derivative as a gradient function.	5	☐
	Estimate the gradient at a point as a limit of gradients of chords.	6	☐

S5.2	Increasing and decreasing functions		
	Book Section 9B	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*	
Graphical interpretation of $f'(x) > 0$, $f'(x) = 0$, $f'(x) < 0$.	Identify intervals on which a function is increasing ($f'(x) > 0$) and decreasing ($f'(x) < 0$).	7	☐
	Sketch the graph of the derivative from the graph of a function.	8	☐
	Sketch the graph of a function from the graph of its derivative.	9	☐

S5.3	Derivatives of polynomials		
	Book Section 9C	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*	
The derivative of the functions of the form $f(x) = ax^n + bx^{n-1} + \cdots$ where all exponents are integers.	Apply the rule to differentiate polynomials using: $f(x) = x^n \Rightarrow f'(x) = nx^{n-1}$	10	☐
	Rearrange an expression into the form: $f(x) = ax^n + bx^{n-1} + \cdots$ before differentiating.	11	☐

S5.4	Equations of tangents and normals		
	Book Section 9D	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*	
Tangents and normals at a given point, and their equations.	Evaluate the gradient at a given point.	12	☐
	Find the point on the curve with a given gradient.	13	☐
	Find the equation of the tangent to the curve $y = f(x)$ at the point (x_1, y_1) using: $y - y_1 = m(x - x_1)$ where $y_1 = f(x_1)$ and $m = f'(x_1)$.	14	☐
	Find the equation of the normal to the curve using: $y - y_1 = -\frac{1}{m}(x - x_1)$	15	☐
	Solve problems involving tangents and normal.	16	☐
	Use technology to find the gradient and the equation of the tangent at a given point.	17	☐
	Use technology to draw the graph of the gradient function.	18	☐

S5.5

Introduction to integration		
Book Section 10A, 10B	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*
Integration as anti-differentiation of functions of the form $f(x) = ax^n + bx^{n-1} + \cdots$, where $n \in \mathbb{Z}$, $n \neq -1$.	Use $\int ax^n \, dx = \frac{a}{n+1}x^{n+1} + c$, for $n \neq -1$.	19 ☐
	Rearrange an expression into the form: $f(x) = ax^n + bx^{n-1} + \cdots$ before integrating.	20 ☐
Definite integrals using technology. Area of a region enclosed by a curve $y = f(x)$ and the x-axis, where $f(x) > 0$.	Use technology to evaluate integrals of the form: $\int_a^b f(x)\, dx$, and interpret this as the area between the curve and the x-axis.	21 ☐
Anti-differentiation with a boundary condition to determine the constant term.	Find the expression for y in terms of x when given $\frac{dy}{dx}$ and one pair of (x, y) values.	22 ☐

S5.6

Local maximum and minimum points		
Book Section 16A	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*
Values of x where the gradient of the curve is zero. Solving $f'(x) = 0$.	Use the expression for $\frac{dy}{dx}$ to find points where the gradient is zero.	23 ☐
	Use technology to sketch the graph of $f'(x)$ and solve $f'(x) = 0$.	24 ☐
Local maximum and minimum points.	Locate local maximum and minimum points on a graph.	25 ☐
	Be aware that the greatest/least value of a function may occur at an end-point of the domain.	26 ☐

S5.7

Optimization problems in context		
Book Section 16B	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*
Optimization.	Find the maximum or minimum value of a function in a real-life context.	27 ☐

S5.8

The trapezoidal rule		
Book Section 16C	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*
Approximating areas using the trapezoidal rule.	Estimate area given a table of data.	28 ☐
	Estimate the area given a function.	29 ☐

H5.9

More differentiation rules		
Book Section 10A, 10B, 10C, 10D	Revised ☐	
Syllabus wording	*You need to be able to:*	*Question*
Derivative of $\sin x$, $\cos x$, e^x, $\ln x$, $x^n (n \in \mathbb{Q})$.	Apply the rules of differentiation to these functions.	30 ☐
The chain rule, product rule and quotient rules.	Use $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$.	31 ☐
	Use $\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$.	32 ☐
	Use $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{du}{dx}}{v^2}$	33 ☐
Related rates of change.	Use the chain rule to find related rates of change.	34 ☐

H5.10 The second derivative

Book Section 10E | Revised ☐

Syllabus wording	You need to be able to:	Question
The second derivative.	Find the second derivative, and understand the notation $f''(x)$ and $\frac{d^2y}{dx^2}$.	35 ☐
Use of second derivative test to distinguish between a maximum and a minimum point.	Use the second derivative to distinguish between maximum ($f''(x) < 0$) and minimum ($f''(x) > 0$).	36 ☐
	Use the terms 'concave up' for $f''(x) > 0$ and 'concave down' for $f''(x) < 0$.	37 ☐
	Describe sections of a graph as 'concave up' or 'concave down'.	38 ☐
	Be aware that a point of inflection is a point at which the concavity changes and interpret this in context.	39 ☐

H5.11 More integration techniques

Book Section 11A, 11B | Revised ☐

Syllabus wording	You need to be able to:	Question
Definite and indefinite integration of x^n where $x \in \mathbb{Q}$, including $n = -1$, $\sin x$, $\cos x$, $\frac{1}{\cos^2 x}$ and e^x.	Use the rules for finding indefinite integrals of these functions.	40 ☐
	Evaluate definite integrals by substituting the limits into the integrated expression.	41 ☐
Integration by inspection, or substitution of the form $\int f(g(x))g'(x)dx$.	When integrating $f(ax + b)$, remember to divide by a.	42 ☐
	Recognize instances of 'reverse chain rule'.	43 ☐

H5.12 Area between curve and the *y*-axis and volumes of revolution

Book Section 11C | Revised ☐

Syllabus wording	You need to be able to:	Question
Area of the region enclosed by a curve and the x or the y axes in a given interval.	Know that the integral has a negative value if the area is under the x-axis, and use $\int_a^b \lvert f(x) \rvert dx$ to find the area enclosed by the curve and the x-axis.	44 ☐
	Find the area between the curve and the y-axis by using: $\int_c^d g(y)dy$.	45 ☐
Volumes of revolution about the x-axis or the y-axis.	Use $V = \int_a^b \pi y^2\, dx$ for rotation about the x-axis.	46 ☐
	Use $V = \int_c^d \pi x^2\, dy$ for rotation about the y-axis.	47 ☐

H5.13 Derivatives and integrals in kinematics

Book Section 12A | Revised ☐

Syllabus wording	You need to be able to:	Question
Kinematic problems involving displacement s, velocity v and acceleration a.	Use differentiation: $v = \frac{ds}{dt}$ and $a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$.	48 ☐
	Know that speed is the magnitude of velocity.	49 ☐
	Understand the notation $\dot{x} = \frac{dx}{dt}$ and $\ddot{x} = \frac{d^2x}{dt^2}$.	50 ☐
	Use $a = v\frac{dv}{dx}$.	51 ☐
	Calculate the change of displacement using $\int_{t_1}^{t_2} v(t)dt$.	52 ☐
	Calculate total distance travelled using $\int_{t_1}^{t_2} \lvert v(t) \rvert dt$.	53 ☐

H5.14 Differential equations and separation of variables

Book Section 13A | Revised ☐

Syllabus wording	You need to be able to:	Question
Setting up a mode/differential equation from a context.	Write down a differential equation from a word problem.	54 ☐
Solving by separation of variables.	If $\frac{dy}{dx} = f(x)g(y)$ then $\int \frac{1}{g(y)} dy = \int f(x)dx$	55 ☐
	Know that the general solution contains one unknown constant.	56 ☐
	Find the constant by using the initial condition.	57 ☐

H5.15	Slope fields	
	Book Section 13B	Revised ☐
Syllabus wording	*You need to be able to:*	*Question*
Slope fields and their diagrams.	Draw a slope field for a given differential equation.	58 ☐
	Interpret a given slope field.	59 ☐

H5.16	Euler's method for finding approximate solutions to differential equations	
	Book Section 13B	Revised ☐
Syllabus wording	*You need to be able to:*	*Question*
Numerical solution of $\frac{dy}{dx} = f(x, y)$.	For a given step size h, use: $x_{n+1} = x_n + h$, $y_{n+1} = y_n + hf(x_n, y_n)$.	60 ☐
Numerical solution of the coupled system $\frac{dx}{dt} = f_1(x, y, t)$, $\frac{dy}{dt} = f_2(x, y, t)$.	For a given step size h, use: $t_{n+1} = t_n + h$ $x_{n+1} = x_n + f_1(x_n, y_n, t_n)$, $y_{n+1} = y_n + f_2(x_n + y_n, t_n)$.	61 ☐

H5.17	Coupled systems and phase portraits	
	Book Section 13C	Revised ☐
Syllabus wording	*You need to be able to:*	*Question*
Phase portrait for the solutions of coupled differential equations of the form $\frac{dx}{dt} = ax + by$, $\frac{dy}{dt} = cx + dy$.	Know that the behaviour of the system depends on the eigenvalues of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.	62 ☐
Qualitative analysis of future paths for distinct, real, complex and imaginary eigenvalues. Sketching trajectories.	If the eigenvalues are positive, or complex with positive real part, all solutions move away from the origin. If the eigenvalues are negative, or complex with negative real part, all solutions move towards the origin.	63 ☐
	If the eigenvalues are complex, the solutions form a spiral. The direction of the spiral can be determined by looking at the sign of $\frac{dx}{dt}$ when $x = 0$. If the eigenvalues are imaginary, the solutions form a circle or ellipse.	64 ☐
	If the eigenvalues are real with different signs, the origin is a saddle point.	65 ☐
Using phase portraits to identify key features such as equilibrium points, stable populations and saddle points.	A solution is called stable if solutions tend towards it; otherwise it is called unstable. A saddle point has solution curves approaching it along one line and moving away along another. All of these are called equilibrium points.	66 ☐
Find the exact solution of the system in the case of real distinct eigenvalues.	The general solution is $\begin{pmatrix} x \\ y \end{pmatrix} = Ae^{\lambda_1 t}\mathbf{v_1} + Be^{\lambda_2 t}\mathbf{v_2}$ where λ_1, λ_2 are the eigenvalues and $\mathbf{v_1}, \mathbf{v_2}$ eigenvectors.	67 ☐

H5.18	Second order differential equations	
	Book Section 13D	Revised ☐
Syllabus wording	*You need to be able to:*	*Question*
Solutions of $\frac{d^2x}{dt^2} = f\left(x, \frac{dx}{dt}, t\right)$ by Euler's method.	Re-write the system as coupled first order equations: $\frac{dx}{dt} = y$, $\frac{dy}{dt} = f(x, y, t)$	68 ☐
	Sketch a phase portrait for an equation of the form: $\frac{d^2x}{dt^2} + a\frac{dx}{dt} + bx = 0$	69 ☐
	Use eigenvalues and eigenvectors to find an exact solution for an equation of the form: $\frac{d^2x}{dt^2} + a\frac{dx}{dt} + bx = 0$	70 ☐

Practice questions

1 In this question, x is measured in degrees. Use a table to estimate, to two decimal places, the limit of $\frac{\sin 3x}{0.2x}$ when x tends to zero.

2 Use a graph to estimate the limit of $\frac{\ln\left(\frac{x}{2}\right)}{x-2}$ when x tends to 2.

3 Given that $y = 3x^2 - 5x$ and $\frac{dy}{dx} = 6x - 5$, what is the value of the derivative of y when $x = 2$?

4 Write an equation to represent the following situation:
The area decreases with time at a rate proportional to the current area.

5 The table shows some information about a function f(x).

x	1	3	4
f(x)	4	8	5
f'(x)	–1	4	2

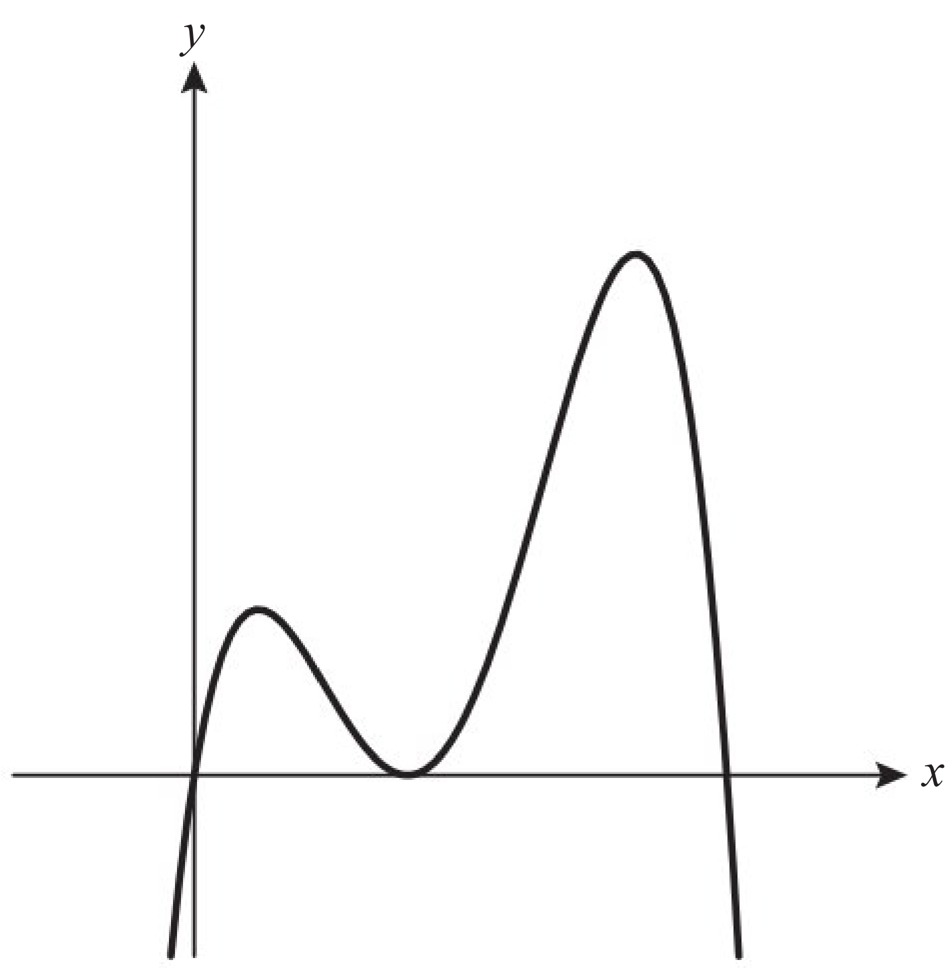
A graph has equation $y = f(x)$. Find the gradient of the graph at the point where $y = 4$.

6 Point $P(4, 2)$ lies on the curve with equation $y = \sqrt{x}$. The table shows the coordinates of a variable point Q and the gradient of the chord PQ. Complete the table and use it to estimate the gradient of the curve at P.

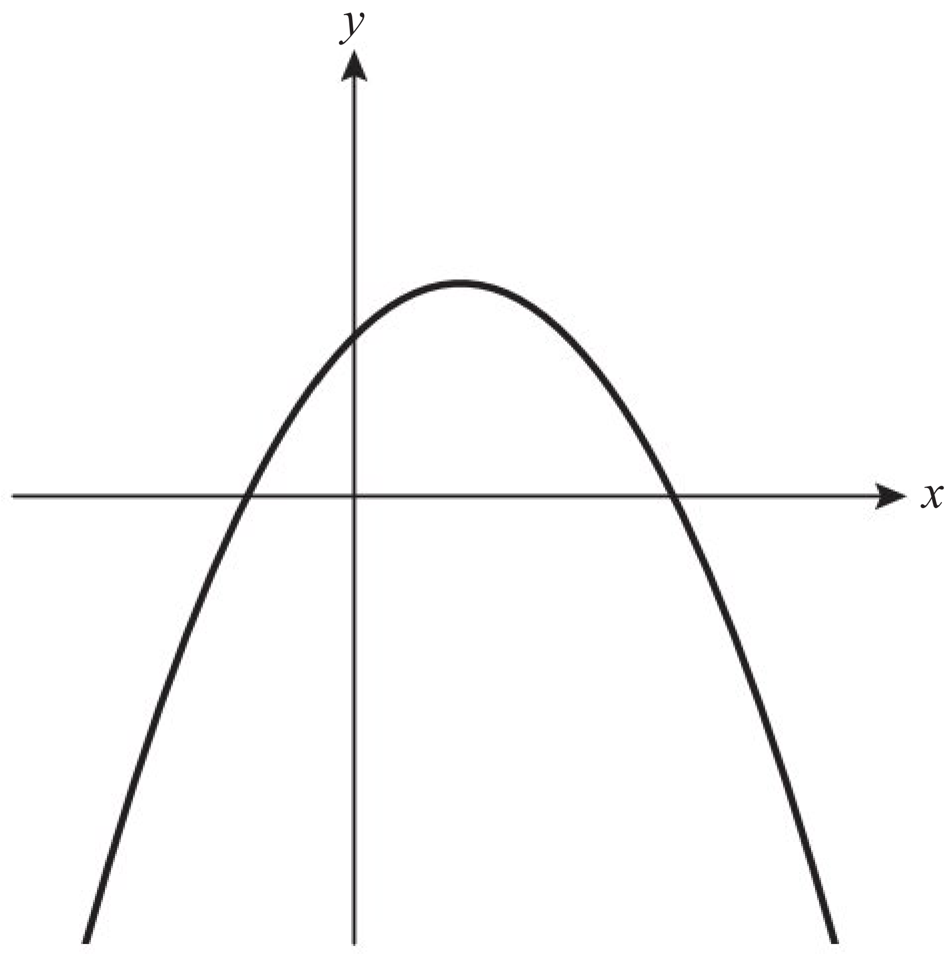
x_Q	y_Q	Δx	Δy	Gradient of *PQ*
5	2.236	1	0.236	0.236
4.1	2.025	0.1		
4.01				
4.001				

7 Use technology to sketch the graph of $f(x) = x^3 - 5x + 2$ and use it to find the range of values of x for which $f'(x) < 0$.

8 The graph of $y = f(x)$ is shown here. Sketch the graph of $y = f'(x)$.



9 The graph of $y = f'(x)$ is shown here. Sketch one possible graph of $y = f(x)$.



10 Differentiate $y = 4x^2 - \frac{1}{10}x^{-5} - 3x + 2$.

11 Find $f'(x)$ when:

a $f(x) = 3x^2(4 - x^4)$

b $f(x) = 1 - \frac{3}{2x^4}$

c $f(x) = \frac{4x^2 - 3x + 1}{5x}$

12 Given that $f(x) = 4x^2 - 2x^{-1}$, find $f'(x)$ and evaluate $f'(2)$.

13 For the curve with equation $y = 12x + 5x^{-1}$, find $\frac{dy}{dx}$.
Hence find the x-coordinates of the points on the curve $y = 12x + 5x^{-1}$ where the gradient equals 2.

14 A curve has equation $y = x^2 - 3$. Find the equation of the tangent to the curve at the point where $x = 4$.

15 Find the equation of the normal to the curve $y = 3x - 2x^{-1}$ at the point where $x = 2$.

16 The tangent to the curve with equation $y = x^2 - 3$ at the point (a, b) passes through $(0, -12)$. Find the possible values of a.

17 For the curve with equation $y = \frac{4\sqrt{x} - 3}{7x^2}$ find, correct to two decimal places:

a the gradient when $x = 3.2$

b the equation of the tangent at the point where $x = 3.2$.

18 A curve has equation $y = \frac{4\sqrt{x} - 3}{7x^2}$. Find the coordinates of the point on the curve where the gradient is 2.

19 Find $\int 9\,x^2 + 6x^{-3}\,dx$.

20 Find $\int \frac{x^5 - 3}{2x^2}\,dx$.

21 Find the area enclosed by the curve $y = 2x^3 - 1$, the x-axis and the lines $x = 2$ and $x = 3$.

22 Given that $\frac{dy}{dx} = 4x + 2$, and that $y = 3$ when $x = 2$, find an expression for y in terms of x.

23 A curve has equation $y = 2x^3 - ax^2 + 3$. Find, in terms of a, the x-coordinates of the points where the gradient of the curve is zero.

24 Given that $f(x) = \frac{2}{x} + \sqrt{x}$, use the graph of $y = f'(x)$ to solve the equation $f'(x) = 0$.

25 Find the coordinates of the local maximum point on the graph of $y = 2x^3 - 0.4x^2 - 0.7x + 2$.

26 Find the smallest value of the function $f(x) = 0.1x^5 - 2x^3$ in the interval $-5 \leq x \leq 5$.

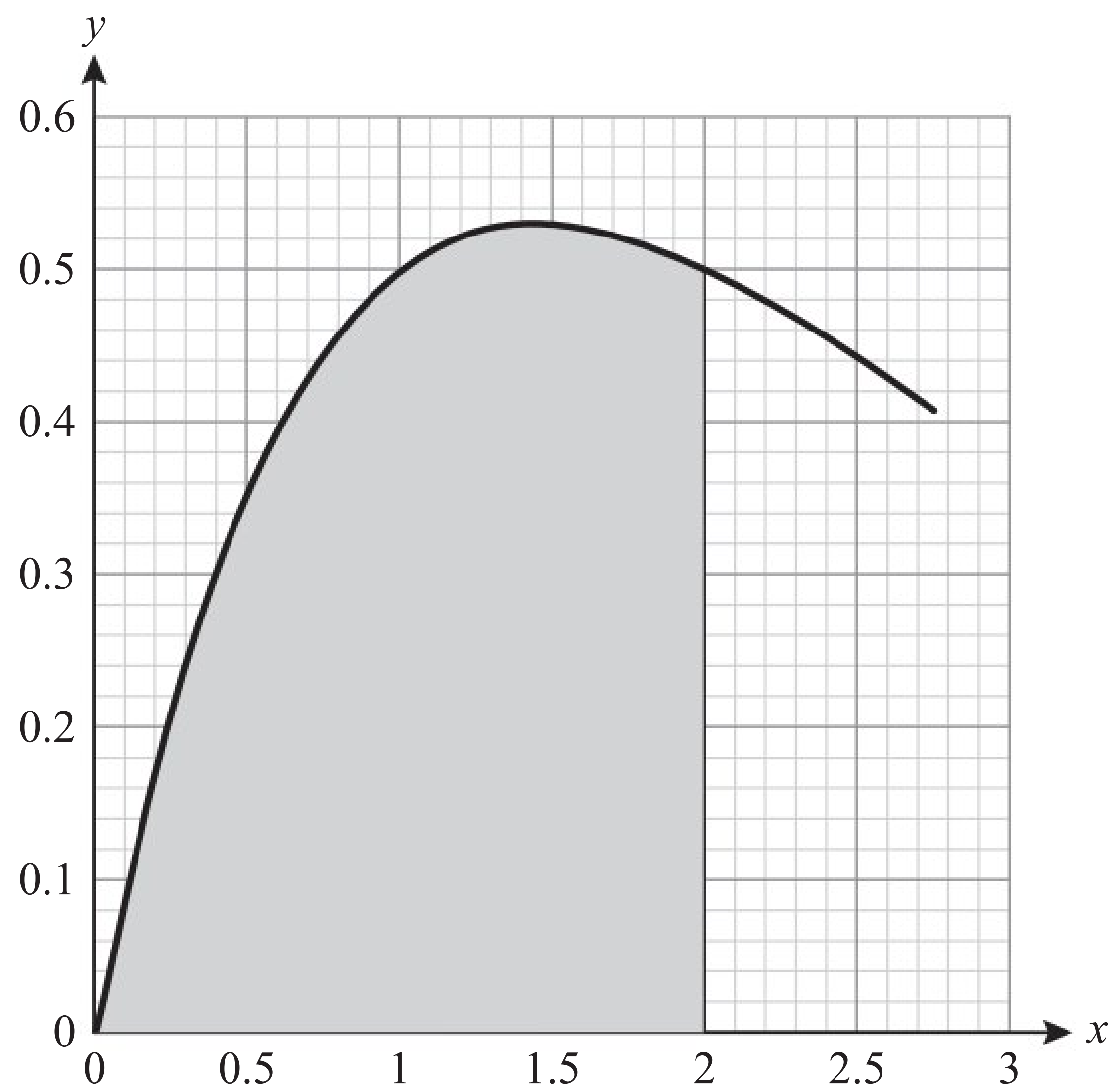
27 An open box has a square base of side x cm and height $\frac{32}{x^2}$ cm. Show that the surface area of the box is given by $S = x^2 + \frac{128}{x}$, and find the minimum possible surface area of the box.

28 Some of the values of the function f(x) are given in the table:

x	2	2.5	3	3.5	4	4.5	5
f(x)	1.6	2.1	2.3	2.2	2.0	1.5	0.8

Use all the values in the table to estimate the value of $\int_2^5 f(x)\, dx$.

29 The diagram shows a part of the graph of $y = \frac{x}{2^x}$.



Use the trapezoidal rule with five strips to estimate the shaded area.

30 Differentiate $3 \sin x - 5 \cos x + 2$.

31 Differentiate:

a $\sqrt{3x^2 - 1}$

b $2 \sin^3(5x)$.

32 Given that $y = 4xe^{-3x}$, find $\frac{dy}{dx}$.

33 Given that $f(x) = \frac{\ln x}{4x}$, show that $f'(x) = \frac{1 - \ln x}{4x^2}$.

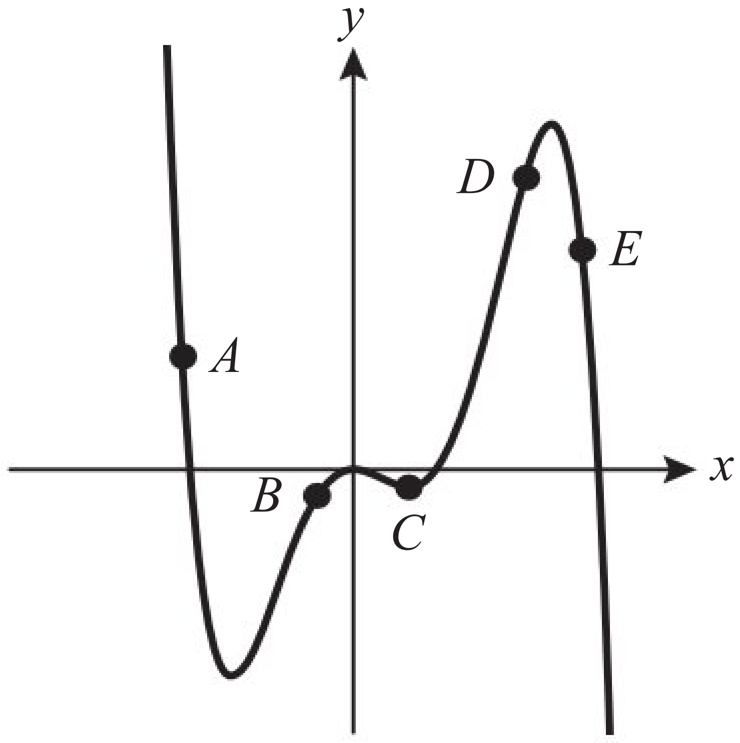
34 The area of a circle decreases at a rate of $3\,cm^2\,s^{-1}$.
Find the rate at which the radius of the circle is decreasing at the time when it equals 12 cm.

35 Given that $y = x^3 - 3 \ln x$, find $\frac{d^2y}{dx^2}$.

36 Show that the graph of $f(x) = \sin x - \cos x$ has a local maximum point at $\left(\frac{3\pi}{4}, \sqrt{2}\right)$.

37 Find the range of values of x for which the function $f(x) = 5x^3 - 2x^2 + 1$ is concave up.

38 For the following graph, write down the points at which the function is concave down.



39 Find the x-coordinate of the point of inflection on the curve with equation $y = 3x^5 - 10x^4 + 8x + 2$.

40 Find $\int 2x^{-\frac{2}{3}} + \frac{4}{3x}\,dx$.

41 Evaluate $\int_0^{\frac{\pi}{6}} \sin 2x\,dx$.

42 Find $\int \left(2e^{4x} + 3e^{-\frac{1}{3}x}\right)dx$.

43 Find the following integrals:

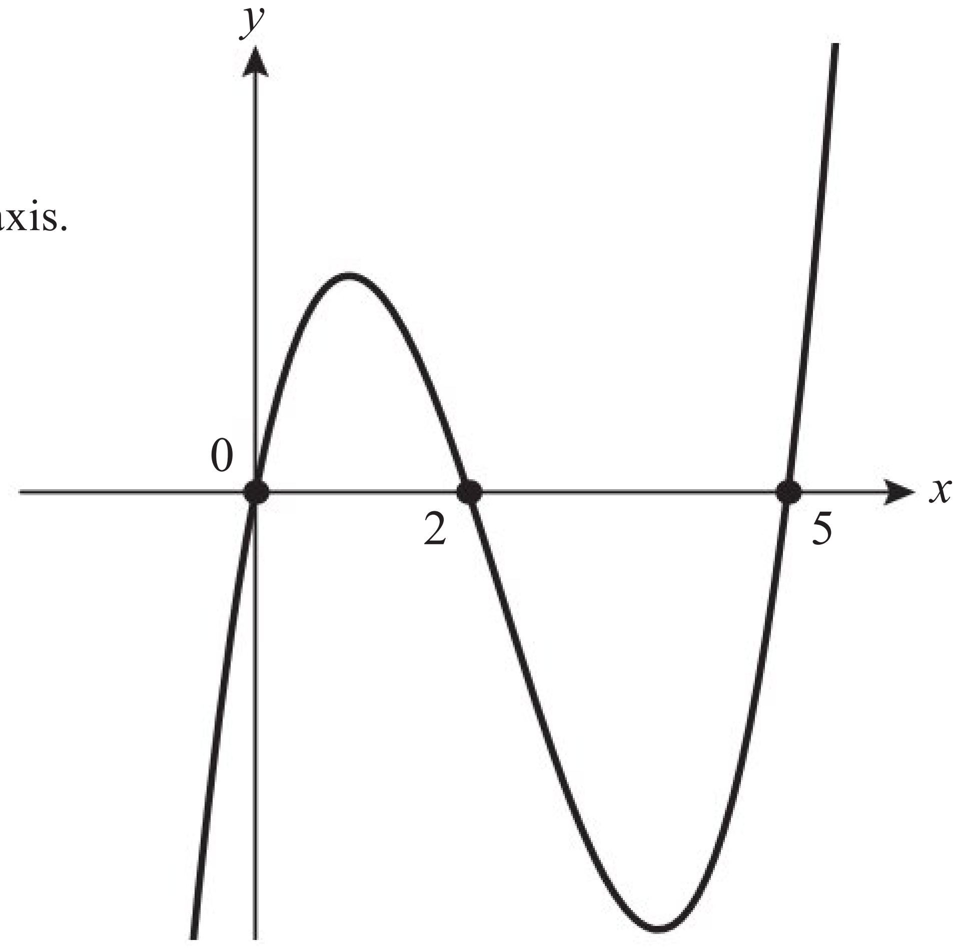
a $\int 4\cos x \sin^2 x\,dx$

b $\int \frac{x}{x^2+3}\,dx$

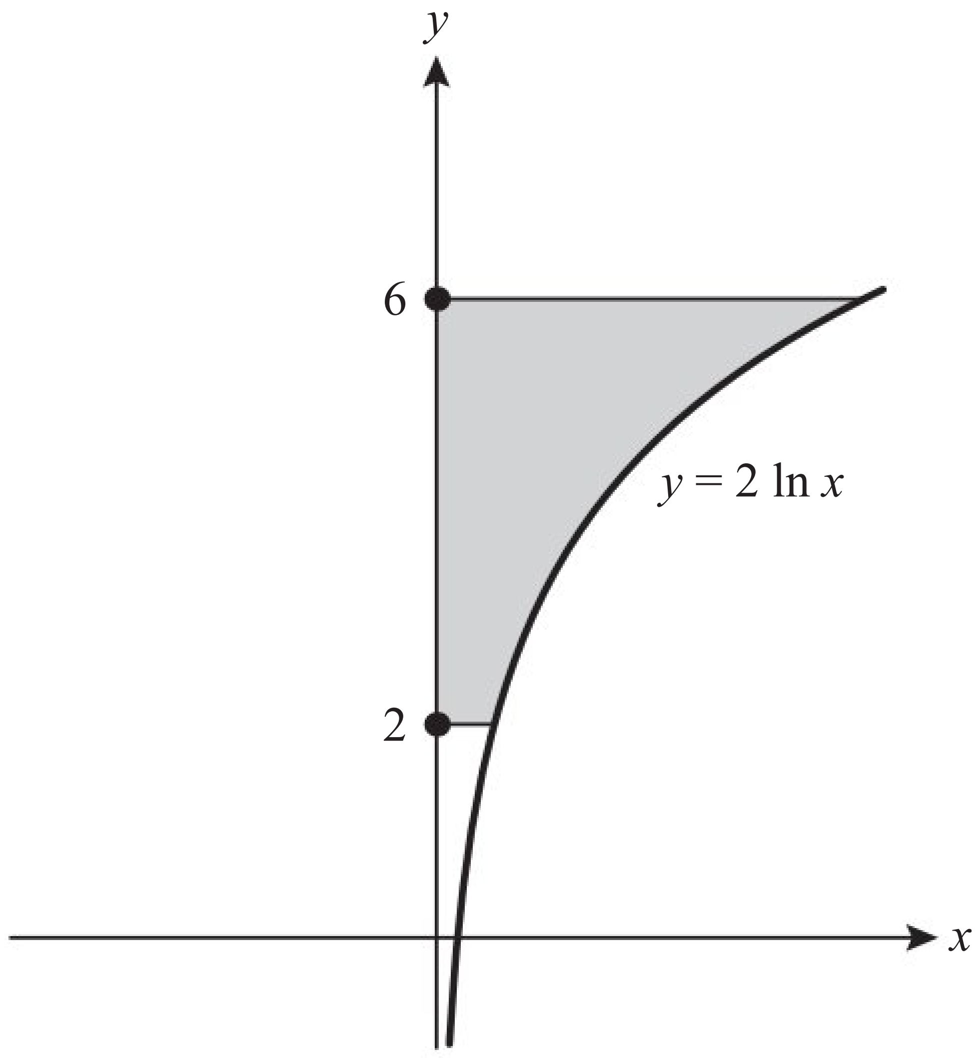
44 The diagram shows the graph of $f(x) = x^3 - 7x^2 + 10x$.

a Find $\int_2^5 f(x)\,dx$.

b Find the area enclosed by the curve and the x-axis.



45 The curve in the diagram has equation $y = 2\ln x$.
Find the exact value of the shaded area.



46 The region enclosed by the curve $y = \sin x$ and the x-axis, between $x = 0$ and $x = \pi$, is rotated around the x-axis. Find the volume of the resulting solid of revolution.

47 The part of the curve $y = x^2$ between $x = 1$ and $x = 3$ is rotated around the y-axis.
Find the volume of the resulting solid.

48 The displacement, s m, of an object at time t seconds, is given by $s = 3\sin(5t)$.
Find the acceleration of the object after 2 seconds.

49 The displacement of an object is given by $s = 3e^{-0.2t}$, where s is measured in metres and t in seconds.
Find the speed of the object after 4 seconds.

50 The displacement of a particle at time t is x. Given that $\dot{x} = 3t^2 - 2t$, find the acceleration of the particle at time t.

51 The displacement of an object is denoted by x and its velocity by v. Given that $v = 4\sin x$, find the acceleration when the displacement is 2.

52 The velocity of an object, measured in m s^{-1}, is given by $v = \frac{1}{\sqrt{t+3}}$. When $t = 2$, the displacement of the object from the origin is 4 m.
Find the displacement from the origin when $t = 5$.

53 The velocity of an object at time t seconds is given by $v = 2\cos(0.4t)\,\text{m s}^{-1}$.
Find the distance travelled by the object in the first 10 seconds.

54 The rate of decrease of area covered by forest is proportional to the cube root of the area.
Write this as a differential equation.

55 Find the general solution of the differential equation:

$$\frac{dy}{dx} = xy^2 + x$$

56 Find the general solution of the differential equation:

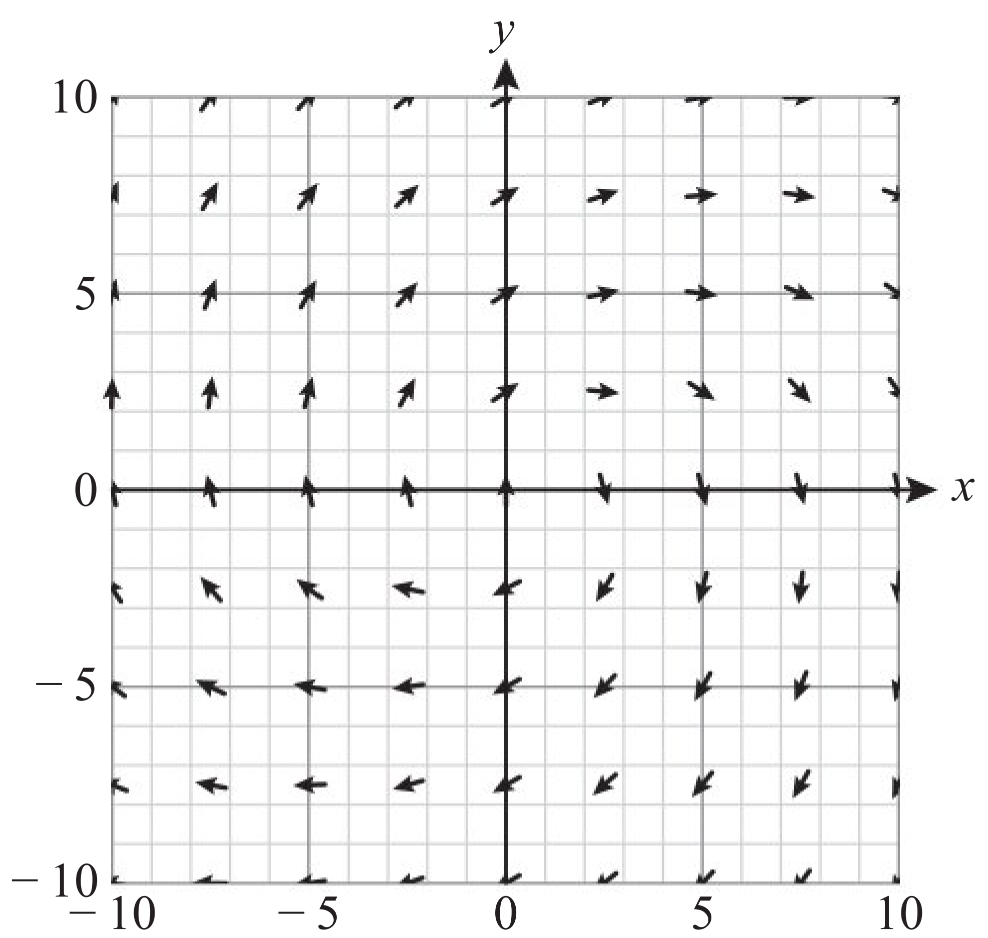
$$\frac{dy}{dx} = 4xy^2$$

Give your answer in the form $y = f(x)$.

57 Solve the differential equation $\frac{dy}{dx} = (x - 1)(y + 2)$ given that $y = 1$ when $x = 1$.
Express y in terms of x.

58 For the differential equation $\frac{dy}{dx} = \frac{y}{x}$, sketch the slope field at points (x, y) with $x, y \in \{1, 2, 3\}$.

59 The diagram shows a slope field for a differential equation. Sketch the trajectory passing through (1, 2).



60 Variables x and y satisfy the differential equation $\frac{dy}{dx} = \sin(x + y)$. When $x = 0$, $y = 2$.
Use Euler's method with step size 0.1 to approximate the value of y when $x = 0.4$. Give your answer to three decimal places.

61 A system of differential equations is given by:
$\frac{dx}{dt} = 2xt + y$, $\frac{dy}{dt} = x + 2y$
When $t = 0$, $x = 0$ and $y = 2$. Use Euler's method with step size 0.1 to find approximate values of x and y when $t = 0.5$.

62 The system of differential equations

$$\frac{dx}{dt} = x - 3.5y,\ \frac{dy}{dt} = 14x + y$$

can be written in the matrix form as

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix}.$$

Find the eigenvalues of the matrix A.

63 Sketch the phase portrait for the system of equations:

$$\begin{cases} \dot{x} = x - 2y \\ \dot{y} = 4x - 5y \end{cases}$$

64 Sketch the phase portrait for the coupled differential equations

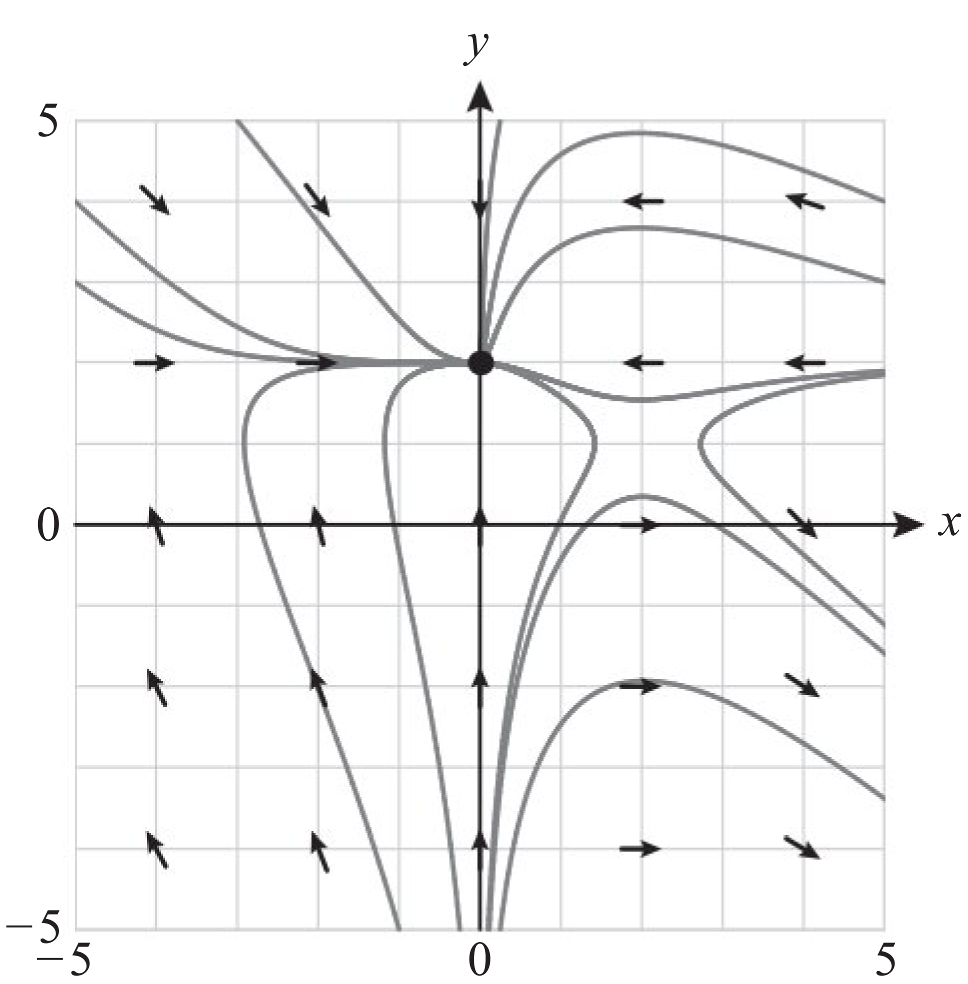
$$\frac{dx}{dt} = x - 3.5y,\ \frac{dy}{dt} = 14x + y$$

Indicate clearly the direction of the arrows.

65 Sketch the phase portrait for the following system of differential equations:

$$\begin{cases} \dot{x} = 2x + 2y \\ \dot{y} = 5x - y \end{cases}$$

66 For this phase portrait, state the coordinates of the equilibrium points. For each equilibrium point, state whether it is stable, a saddle, or other unstable.



67 Find the general solution of the following system of differential equations:

$$\begin{cases} \dot{x} = 2x + 2y \\ \dot{y} = 5x - y \end{cases}$$

68 Consider the differential equation:

$$\frac{d^2x}{dt^2} = 2x\frac{dx}{dt}$$

With $x = 1$, $\frac{dx}{dt} = 2$ when $t = 0$.

Use Euler's method with step length 0.05 to estimate the value of x when $t = 0.2$

69 Sketch a phase portrait for the differential equation

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = 0$$

70 Find the general solution of the differential equation

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = 0$$

Paper plan

	Syllabus Section	Description	Book Section	Mastery: Covered	Mastery: Revised	Mastery: Confident	Practice Paper Coverage: Book P1	Book P2	Book P3	A1	A2	A3	B1	B2	B3	C1	C2	C3
Number and algebra Core	S1.1	Standard form	1B							2a						13		
	S1.2	Arithmetic sequences and series	2A				4							1a		1		
	S1.3	Geometric sequences and series	2B				8			9				1b				
	S1.4	Financial applications of geometric sequences	2C				8				1			1				
	S1.5a	Exponents with integer coefficients	1A														6b	
	S1.5b	Introduction to logarithms	1C										11					
Number and algebra SL	S1.6	Approximation and estimation	11A				13c	6d					3c					2
	S1.7	Finance: Amortization and annuities	11B								1							
	S1.8a	Solving systems of equations	12A							4a	13					14d		
	S1.8b	Solving polynomial equations	12B							4c								
Number and algebra HL	H1.9	Laws of logarithms	1B				2				4b							
	H1.10	Rational exponents	1A				7a										6e	
	H.11	Infinite geometric sequences	1C							9						3b		
	H1.12	Cartesian form of complex numbers	6A				6, 10a			13					2	5		
	H1.13a	Modulus-argument and exponential form of complex numbers	6B				19					1	16		2			
	H1.13b	Geometrical interpretation of complex numbers	6C				10b			13					2			
	H1.14a	Definition and arithmetic of matrices	3A									2					4a	
	H1.14b	Determinants and inverses	3B									2		6e			4bc	
	H1.14c	Solutions of systems of equations using the inverse matrix	3C														4bc	
	H1.15	Eigenvectors and eigenvalues	3D					3cd				2		6cd, 4a		17b	4de	
Functions Core	S2.1	Equation of a straight line	4A							15a			14b					2
	S2.2	Concept of a function	3A				5a				2cd							
	S2.3	Graph of a function	3B													6b		
	S2.4	Key features of graphs	3B							8, 10a	2cd, 4, 6e					9	1bc	

	Syllabus Section	Description	Book Section	Mastery: Covered	Mastery: Revised	Mastery: Confident	Practice Paper Coverage: Book P1	Book P2	Book P3	A1	A2	A3	B1	B2	B3	C1	C2	C3
Functions SL	S2.5a	Linear models	13A							7								
	S2.5b	Quadratic models	13B							4								
	S2.5c	Exponential models	13C				11						11					
	S2.5d	Direct/inverse variation and cubic models	13D				3									13		
	S2.5e	Sinusoidal models	13E									1				16		
	S2.6	Modelling skills	13F								4	1	14c					
Functions HL	H2.7a	Composite functions	5A										7					
	H2.7b	Inverse functions	5B				5b						7			6cd		
	H2.8	Transformations of graphs	5C							8c						6a		
	H2.9	Further modelling	5D				15				4		17					
	H2.10	Using logarithmic scales on graphs	1D							15b								
Geometry and trigonometry Core	S3.1a	Distance and midpoints	4B							4d					2		3ad	
	S3.1b	Volume and surface area of 3D solids	5A					1a								3a		
	S3.1c	Angle between intersecting lines and planes	5B							2								
	S3.2a	Trigonometry in right-angled triangles	5B							2								
	S3.2b	Trigonometry in non right-angled triangles	5B				10b	1bcd					4					
	S3.3	Applications of trigonometry	5C					1		2							3b	
Geometry and trigonometry SL	S3.4	Arcs and sectors of circles	14A					1e					2					
	S3.5	Perpendicular bisectors	14B													11ab	3c	
	S3.6	Voronoi diagrams	14B				12										3cef	

	Syllabus Section	Description	Book Section	Mastery			Practice Paper Coverage											
				Covered	Revised	Confident	Book P1	Book P2	Book P3	A1	A2	A3	B1	B2	B3	C1	C2	C3
Geometry and trigonometry HL	H3.7	Radian measure	4A									1	2					
	H3.8	Further trigonometry	4B							16a	2cd	1	4					
	H3.9	Matrices as transformations	4C					7									4	
	H3.10	Introduction to vectors	2A								3			5				1
	H3.11	Vector equation of a line	2B					6bc			3		14b			11b		1
	H3.12a	Vector applications to kinematics	2C					6			3			5		11		1
	H3.12b	Motion with variable velocity in 2D	12B							16				5				1
	H3.13	The scalar and vector products	2D				16			16b						8		1
	H3.14	Introduction to graph theory	7A											3ab				
	H3.15a	Adjacency matrices	7A											3a			5ab	
	H3.15b	Using adjacency matrices	7B														5cd	
	H3.15c	Weighted graphs and transition matrices	7C										10	3b			5	
	H3.16a	Moving around a graph	7B											3c				
	H3.16b	Minimum spanning tree algorithms	7D							11				3e				
	H3.16c	Chinese postman problem	7E											3cd				
	H3.16d	Travelling salesman problem	7F														5ef	
Statistics and probability Core	S4.1	Sampling	6A							3b			1			2a		
	S4.2	Statistical diagrams	6C				1			1								
	S4.3	Summary statistics	6B										1	1c	1		1a	
	S4.4	Correlation and regression	6D													2bc	1c	2
	S4.5	Definitions in probability	7A													4b		
	S4.6	Probability techniques	7B				1a	2d		1			5					
	S4.7	Discrete random variables	8A					2bce								4		
	S4.8	Binomial distribution	8B				1b	4c		14b							1d	
	S4.9	Normal distribution	8C					2a					15				1d	

	Syllabus Section	Description	Book Section	Mastery: Covered	Mastery: Revised	Mastery: Confident	Practice Paper Coverage: Book P1	Book P2	Book P3	A1	A2	A3	B1	B2	B3	C1	C2	C3
Statistics and probability SL	S4.10	Spearman's rank correlation coefficient	15C														1b	
	S4.11a	χ^2-tests	15A							3a				2d				
	S4.11b	*t*-test	15B													10		
Statistics and probability HL	H4.12a	Sampling and statistical design	9A							14a				2de				
	H4.12b	More on the χ^2 distribution	9B											2d				
	H4.13	Non-linear regression	9C								4							
	H4.14	Transforming variables and unbiased estimators	8A												1		1e	
	H4.15	Linear combinations of normal variables and the distribution of $\bar{X}$	8B										15		1	12		
	H4.16	Confidence intervals for the mean	9D							6								2
	H4.17	Poisson distribution	8C					4					9			18		
	H4.18a	Hypothesis tests for the mean	9E										12a	2ab				
	H4.18b	Hypothesis tests using the binomial and Poisson distributions	9F					4e		14b								
	H4.18c	Hypothesis tests for correlation coefficients	9F														1b	2
	H4.18d	Type I and II errors	9G					4fg					12b			18		
	H4.19	Markov chains	8D					3						6abefg		14		
Calculus Core	S5.1a	Concept of a limit	9A							12	4a			4d	2		6a	
	S5.1b	Interpretation of derivatives	9A							17								
	S5.2	Increasing and decreasing functions	9B														2b	2
	S5.3	Derivatives of polynomials	9C												1			
	S5.4	Tangents and normals	9D														2c	
	S5.5a	Integration as anti-differentiation	10A										6a					
	S5.5b	Definite integrals and areas using technology	10B				13b			5	4		3b, 7			9		
Calculus SL	S5.6	Maximum and minimum points	16A													7a	6c	
	S5.7	Optimization problems	16B										6b		1			1
	S5.8	Trapezoidal rule	16C				13a						3a					

	Syllabus Section	Description	Book Section	Mastery			Practice Paper Coverage											
				Covered	Revised	Confident	Book P1	Book P2	Book P3	A1	A2	A3	B1	B2	B3	C1	C2	C3
Calculus HL	H5.9a	Extending differentiation	10A				7b			16b, 17		1					2a, 6c	
	H5.9b	The chain rule	10B							17			17a				2a	
	H5.9c	The product and quotient rules	10C				14			10b			17b					
	H5.9d	Related rates of change	10D							17						13b		
	H5.10	The second derivative	10E									1	17b			7b	6d	
	H5.11a	Further indefinite integration	11A				7c			12						15	2f	
	H5.11b	Definite integration and area between curve and x-axis	11B														2d	
	H5.12	Area between curve and the y-axis and volumes of revolution	11C				9			5							2e	
	H5.13	Derivatives and integrals in kinematics	12A				14				5		8			9		
	H5.14	Differential equations and separation of variables	13A				18			12				4e	2	15		
	H5.15	Slope fields	13B				17						13ab					
	H5.16	Euler's method for first order differential equations	13B				17						13c					
	H5.17	Coupled systems and phase portraits	13C					6c			6bcd		14	4bcde		17		
	H5.18	Second order differential equations	13D					6ab			6a	1				17		

Practice exam papers

Mathematics: applications and interpretation
Higher level
Practice set A: Paper 1

Candidate session number

2 hours

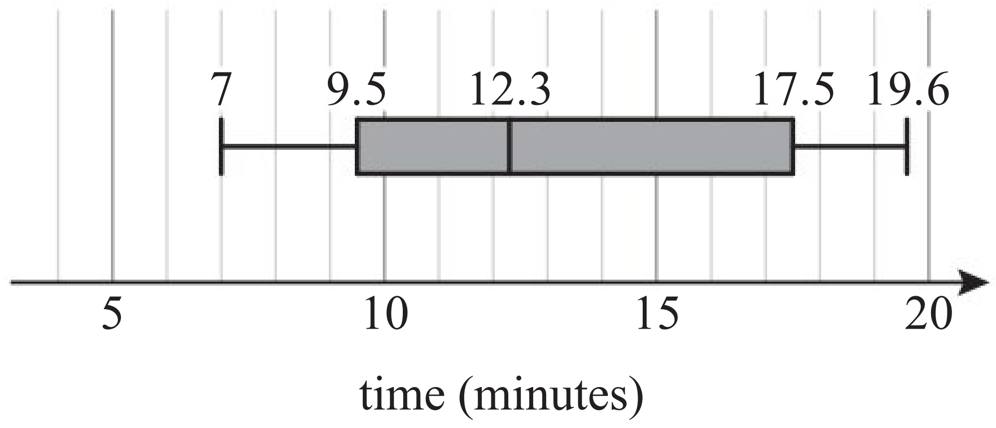
Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer **all** questions. Answers must be written within the answer boxes provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A copy of the mathematics: applications and interpretation formula booklet is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1 *[Maximum mark: 6]*

The box plot summarizes the times taken by a group of 40 children to complete an obstacle course.



Two of the 40 children are selected at random. Find the probability that:

a both completed the course in less than 9.5 minutes *[3]*

b one completed the course in less than 9.5 minutes and the other in between 9.5 and 17.5 minutes. *[3]*

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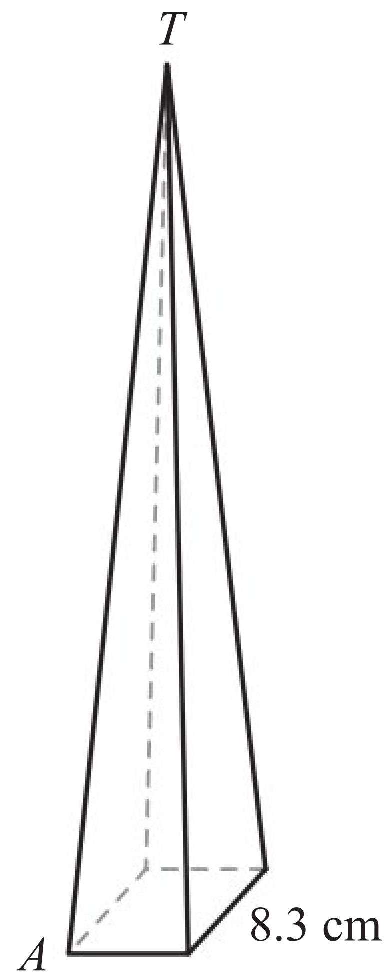
2 *[Maximum mark: 6]*

A flag pole has the shape of a square-based pyramid shown in the diagram. The side length of the base is 8.3 cm. The edge AT makes an angle of 89.8° with the base.

a Find the height of the flag pole. Give your answer in centimetres, in standard form, correct to two significant figures. *[4]*

b A soldier stands 12.5 m from the pole. Find the angle of elevation of the top of the pole. *[2]*



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3 *[Maximum mark: 6]*

Millie is investigating whether teachers and students at her college choose different food at the school cafeteria. She records the choices over several Tuesdays, each of which had the same three options. The results are show in the table.

	Vegetable lasagne	Fishcakes	Sausages and chips
Teachers	17	28	12
Students	87	74	92

a Is there evidence, at the 5% significance level, that meal choice depends on whether a person is a teacher or a student? State your hypotheses and justify your conclusion. *[5]*

b Suggest one improvement that Millie could make to her investigation. *[1]*

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4 *[Maximum mark: 8]*

A ball is projected from a point above ground. A student measures the horizontal and vertical distances of the ball from the starting point and records the results. In the table below, x m is the horizontal distance from the point of projection and y m is the height of the ball above ground.

x	1	2	3
y	7.2	7.4	6.4

The path of the ball is modelled by the equation $y = ax^2 + bx + c$.

a Use the data in the table to find the values of a, b and c, giving your answers correct to two significant figures. *[3]*

b From what height above ground was the ball projected? *[1]*

c Find the value of x when the ball hits the ground. *[2]*

d Find the straight line distance from the point of projection to the point where the ball hits the ground. *[2]*

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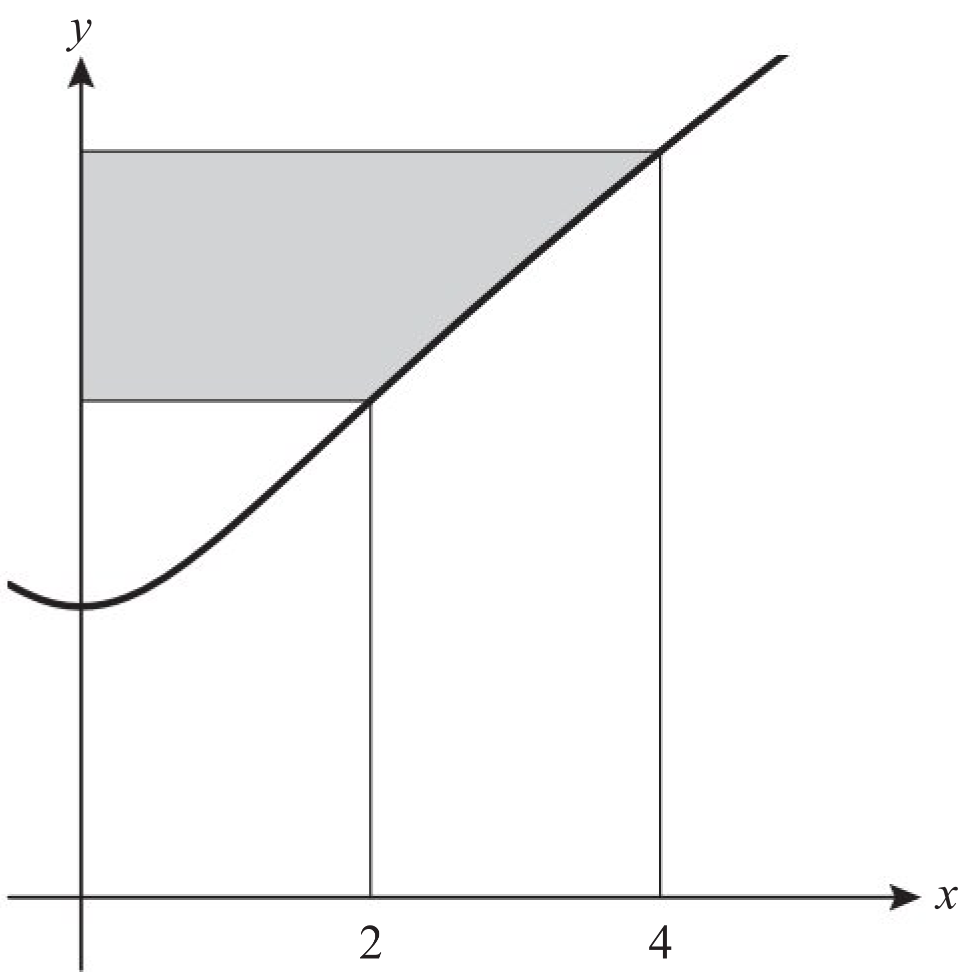
5 *[Maximum mark: 7]*

The curve in the diagram has equation $y = \sqrt[3]{x^2 + 1}$. The shaded region is bounded by the curve, the y-axis and two horizontal lines.



a Find the area of the shaded region. *[4]*

b Find the volume generated when the shaded region is rotated 2π radians about the y-axis. *[3]*

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6 *[Maximum mark: 6]*

Pietro is investigating the weight of loaves of bread sold by the local bakery. He knows that the weights are distributed normally with standard deviation 7.5 grams. He weighs a random sample of seven loaves and obtains the following results (in grams):

789, 812, 806, 797, 800, 802, 799

a Find a 90% confidence interval for the mean weight of the loaves. *[3]*

b Justify your choice of confidence interval. *[1]*

c The bread loaves are sold as weighing 800 g. A customer thinks that the average weight is around 790 g. Comment on the customer's claim in the light of your confidence interval. *[2]*

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7 *[Maximum mark: 9]*

A boat hire company charges $85 plus $5 for every 30 minutes of use.

a Form a linear model for the cost, $C, in terms of the hire time, *t* hours. *[2]*

A second company charges $60 plus $10 for every 30 minutes of use for the first 2 hours, then $15 for every 15 minutes of use thereafter.

b Form a piecewise linear model for the cost, $C, in terms of the hire time, *t* hours. *[4]*

John hires a boat from the first company.

c Find the shortest length of time for which this would prove the cheaper option for him. *[3]*

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8 *[Maximum mark: 8]*

Ilya decorates small cakes in his bakery. The time it takes him to decorate x cakes is modelled by $T(x) = 0.003x^3 + 10x + 200$ minutes.

a If Ilya wants to work at most 12 hours in a day, what is the largest number of cakes he can decorate? *[2]*

b Ilya can choose to decorate between 20 and 40 cakes per day, inclusive. Find the minimum and maximum time he can take per cake on average. *[4]*

Ilya becomes more efficient and is now able to decorate cakes at twice the rate he could before.

c Find the new model in the form $S(x) = ax^3 + bx + c$. *[2]*

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9 *[Maximum mark: 5]*

The sum of the first two terms of a geometric series is 3 and its sum to infinity is 5.

Given that all terms of the series are positive, find the exact common ratio of the series. *[5]*

10 *[Maximum mark: 6]*

The motion of a particle attached to one end of an elastic spring is modelled by the equation $x = 0.5e^{-0.3t}\sin(2.5t)$, where t seconds is the time and x cm is the displacement of the particle from its initial position.

a In subsequent motion, find the number of times the particle will be 0.25 cm from the starting position. *[3]*

b Find an expression for the rate of change of the displacement at time t. *[3]*

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11 *[Maximum mark: 6]*

Use the matrix form of Prim's algorithm, starting at vertex A, to find the minimum spanning tree for the graph represented by the adjacency table below. List the vertices in the order you add them and state the weight of your tree.

	A	B	C	D	E	F
A	–	11	12	25	–	–
B	11	–	14	–	22	18
C	12	14	–	–	24	12
D	25	–	–	–	31	31
E	–	22	24	12	–	35
F	–	18	20	31	35	–

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12 *[Maximum mark: 8]*

The concentration at time t of a reactant, C, in a reaction mixture at fixed temperature is given by the differential equation

$$\frac{dC}{dt} = -kC^2$$

a Show that

$$C = \frac{C_0}{C_0 kt + 1}$$

where C_0 is the initial concentration of the reactant. *[6]*

b Hence find the long-term concentration of the reactant. *[2]*

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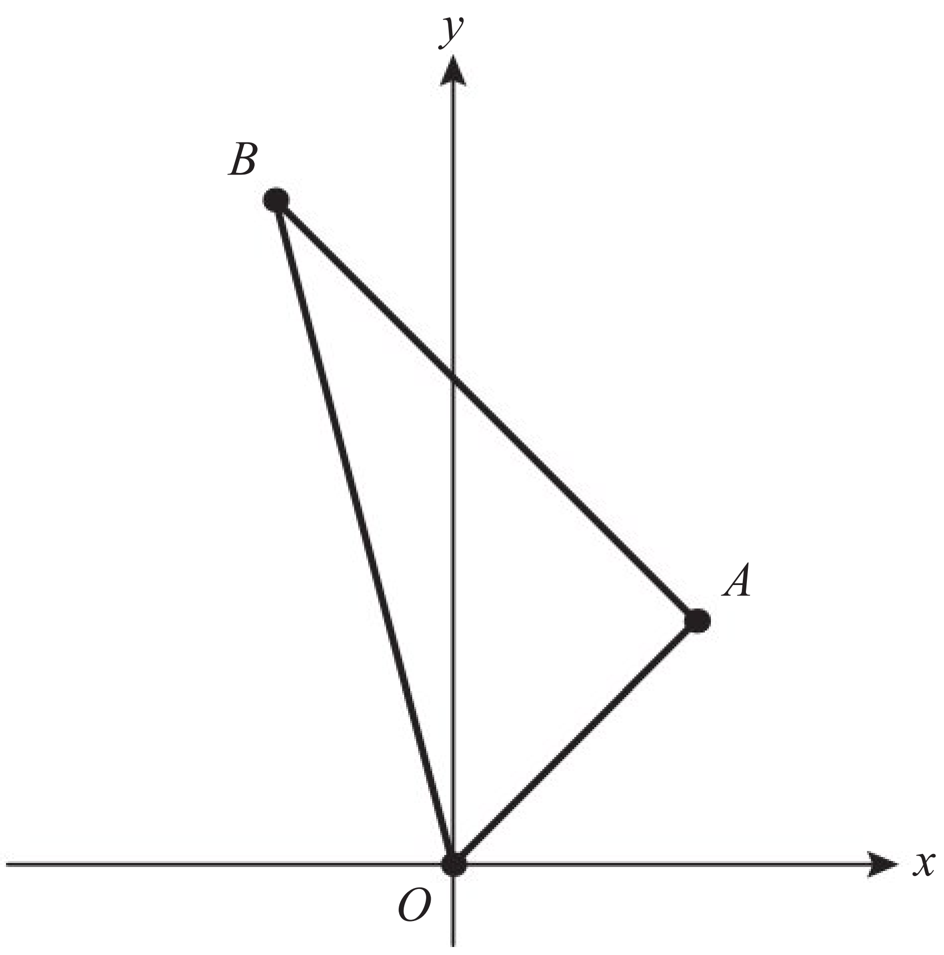
13 *[Maximum mark: 5]*

The diagram shows triangle OAB, with $OB = 2OA$ and angle AOB measuring 60°. Point A has coordinates (3, 3).



a Write down, in Cartesian form, the complex number corresponding to the point A in the Argand diagram. *[1]*

b Find the coordinates of the point B. *[4]*

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14 *[Maximum mark: 6]*

Melinda has read that, 10 years ago, the proportion of households in her town which owned a pet was $\frac{1}{5}$. She suspects that the proportion is now higher. To test her belief, she conducts a survey of 70 randomly selected households and records the number of households, X, which own a pet.

a Melinda decides to knock on doors on a weekday morning and use as her sample the first 70 households who answer the door. State one criticism of this sampling method. *[1]*

b Melinda manages to obtain a simple random sample of 70 households and decides to conduct a test using a 5% significance level. State suitable hypotheses and find the critical region for the test. *[5]*

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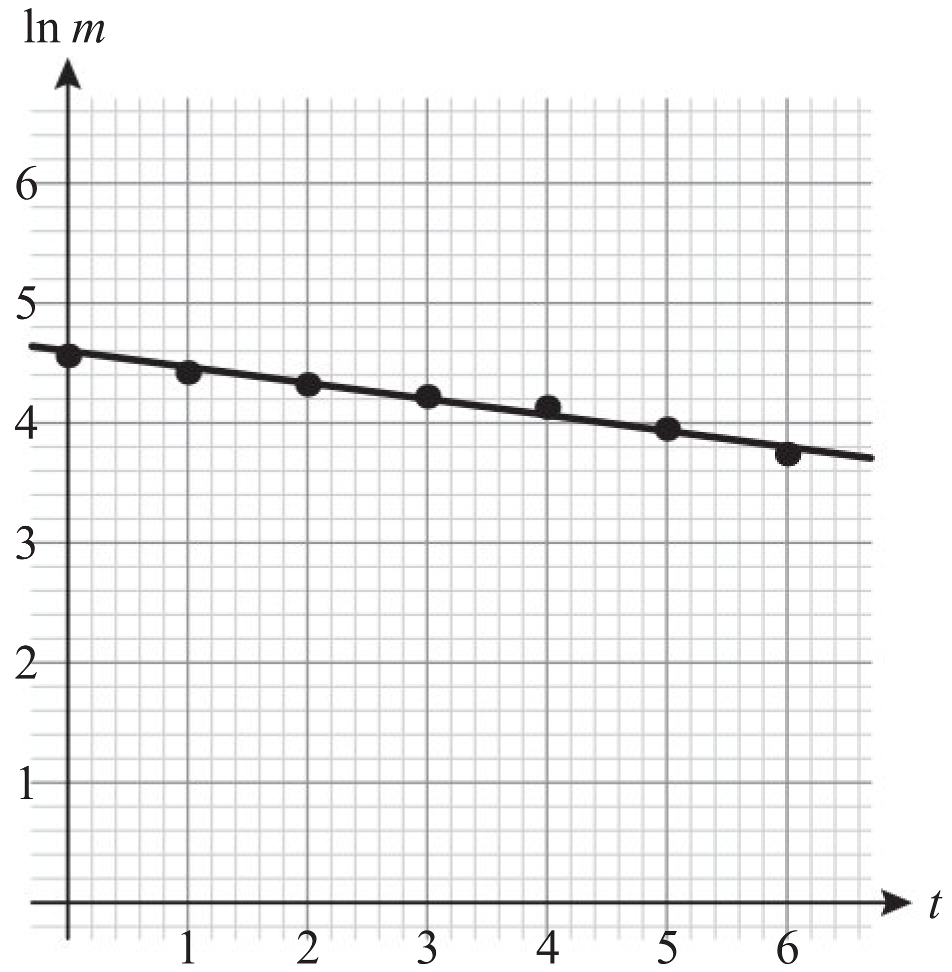
15 *[Maximum mark: 5]*

A student is investigating the rate of decay of caffeine in the bloodstream. A subject drinks a cup of coffee and the amount of caffeine in their bloodstream, m mg, is measured every hour for six hours. The graph below shows the results on a logarithmic scale, with time (t hours) on the horizontal axis and $\ln m$ on the vertical axis. The graph also shows the line of best fit.



a Use the graph to find the equation of the line of best fit. *[3]*

b Hence find an expression for m in terms of t. *[2]*

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16 *[Maximum mark: 7]*

The position vector of an object at time t is given by

$\mathbf{r} = \begin{pmatrix} a \cos kt \\ a \sin kt \end{pmatrix}$ where a > 0

a Show that the object is moving in a circular path. *[3]*

b Show that the velocity vector is perpendicular to the displacement vector. *[4]*

17 *[Maximum mark: 6]*

The total surface area of a cone with radius r and height h is given by $S = \pi r^2 + \pi r\sqrt{r^2 + h^2}$. A cone has height 5 cm and the radius increases at a rate of 2 cm per second.

Find the rate at which the surface area is increasing when the radius equals 10 cm. *[6]*

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Mathematics: applications and interpretation
Higher level
Practice set A: Paper 2

Candidate session number

1 hour 30 minutes

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- A graphical display calculator is required for this paper.
- Answer **all** questions in an answer booklet.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A copy of the mathematics: applications and interpretation formula booklet is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1 *[Maximum mark: 21]*

Suresh is looking to take out a £150 000 mortgage to buy a new house.

He is considering two options:

Mortgage A

10% desposit

2% interest rate compounded annually

25 year repayment period

a **i** Find the annual repayments.

ii Find the total amount he would repay.

iii Find the total amount of interest he would pay. *[7]*

Mortgage B

No deposit

2.5% interest rate compounded monthly

30 year repayment period

b **i** Find the monthly repayments.

ii Find the total amount he would repay. *[5]*

c Explain which mortgage Suresh should choose and why. *[2]*

d Suresh decides to take Mortgage B and invest the money from the 10% deposit. He wants to find an account that will pay a monthly annuity of at least £50 over the lifetime of Mortgage B.

Find the minimum interest rate needed, assuming monthly compounding. *[3]*

e Suresh also saves £250 each month in a regular saver account paying 2% interest (compounded monthly). Show that after n months the balance of the account is

$$a(b^{n-1})$$

where a and b are constants to be found. *[4]*

2 *[Maximum mark: 20]*

Two of the sides of a triangle have length x cm and $2x$ cm, and the angle between them is $\theta°$. The perimeter of the triangle is 10 cm.

a In the case $x = 2$, find the area of the triangle. *[4]*

b Explain why x must be less than $\frac{10}{3}$. *[2]*

c **i** Show that $\cos\theta = \frac{15x - x^2 - 25}{x^2}$

ii Sketch the graph of $y = \frac{15x - x^2 - 25}{x^2}$ for $x > 0$.

iii Hence find the range of possible values of x. *[8]*

d Find the value of x for which the triangle has the largest possible area, and state the value of that area. *[6]*

3 *[Maximum mark: 16]*

A drone D is initially at the point (4, –2, 1) and travels with constant velocity

$\begin{pmatrix} -3 \\ 7 \\ 2 \end{pmatrix} \text{km h}^{-1}$.

A second drone E is initially at the point (–2, 1, –8) and travels with constant velocity

$\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \text{km h}^{-1}$.

a Find the speed of *D*. *[2]*

b Write down equations for position vectors of *D* and *E* at time *t* hours. *[2]*

c **i** Show that the paths of *D* and *E* cross.

ii Find the coordinates of the point at which this occurs. *[5]*

d Show that *D* and *E* do not collide. *[1]*

e **i** Find the time at which *D* and *E* are closest together.

ii Find the minimum distance between *D* and *E*. *[6]*

4 *[Maximum mark: 19]*

It is proposed that the population of sheep, *P* (thousand), on a small island at time *t* years after they were introduced can be modelled by the function

$$P = \frac{5}{1 + Ce^{-kt}}$$

where *C* and *k* are constants.

a Find the long-term size of the population predicted by this model. *[2]*

b Show that $\ln\left(\frac{5}{P} - 1\right) = \ln C - kt$. *[3]*

The following data are collected:

t	2	4	6	8
P	1.9	2.6	3.1	3.7

c Use linear regression to estimate the values of *C* and *k*. *[5]*

d **i** Write down the coefficient of determination for the linear regression.

ii Explain what this suggests about the proposed population model. *[3]*

e Use the model to estimate

i the initial number of sheep introduced to the island

ii the time taken for the population to reach 4500. *[4]*

f Comment on the reliability of your estimates in part **e**. *[2]*

5 *[Maximum mark: 17]*

The velocity (in m s^{-1}) of an object at *t* seconds is given by

$$v(t) = \frac{8 - 3t}{t^2 - 6t + 10}, \ 0 \le t \le 10.$$

Find

a the initial speed *[1]*

b the maximum speed *[2]*

c the length of time for which the speed is greater than 1 m s^{-1} *[3]*

d the time at which the object changes direction *[2]*

e the length of time for which the object is decelerating *[2]*

f the acceleration after 5 seconds *[2]*

g the distance travelled after 10 seconds *[2]*

h the time when the object returns to its starting position. *[3]*

6 *[Maximum mark: 17]*

A small ball is attached to an elastic spring and placed inside a tube filled with viscous liquid. The ball oscillates, with the displacement from its equilibrium position given by the differential equation

$$\frac{d^2x}{dt^2} = -0.06\frac{dx}{dt} - 0.4x$$

The displacement is measured in centimetres and time in seconds. When $t = 2.5$, the ball passes through the equilibrium position with velocity $-3.8\,\text{cm}\,\text{s}^{-1}$.

a By setting $y = \frac{dx}{dt}$, write the differential equation in the form $\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix}$, where **A** is a 2×2 matrix. *[2]*

b Use eigenvalues of **A** to sketch the phase portrait for x and y, justifying the direction of the trajectories. *[6]*

c Use Euler's method with step length 0.05 to find the distance of the ball from the equilibrium position when $t = 3$. *[4]*

The exact solution of the differential equation is $x = e^{-0.03t}(0.0667\sin(0.632t) + 6.48\cos(0.632t))$.

d Comment on the accuracy of your answer from part **c**. *[2]*

e Find the first time after $t = 2.5$ that the ball is instantaneously at rest. Find its distance from the equilibrium position at that time. *[3]*

Mathematics: applications and interpretation
Higher level
Practice set A: Paper 3

Candidate session number

1 hour

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer **all** questions in an answer booklet.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A copy of the mathematics: applications and interpretation formula booklet is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

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1 *[Maximum mark: 25]*

This question is about resonance in vibrating objects.

a Write down the period of the function $\cos \pi t$. *[1]*

b **i** Sketch the function $y = \cos \pi t + \cos 2\pi t$ for $0 \le t \le 3$.

ii Write down the period of the function $\cos \pi t + \cos 2\pi t$. *[2]*

c **i** Use technology to investigate the period of the functions given below. Write down the values of A, B and C.

f(t)	**Period**
$\cos \pi t + \cos 1.5\pi t$	A
$\cos \pi t + \cos 1.25\pi t$	B
$\cos \pi t + \cos 1.1\pi t$	C

ii Hence conjecture an expression for the period, T, of $f(t) = \cos \pi t + \cos\left(\left(1 + \frac{1}{n}\right)\pi t\right)$ where n is an integer. *[4]*

d Prove that, for your conjectured value of T, $f(t + T) = f(t)$. *[3]*

e **i** By considering the real part of $e^{(A+B)i} + e^{(B-A)i}$, find a factorized expression for $\cos(A + B) + \cos(A - B)$.

ii Hence find a factorized form for the expression $\cos P + \cos Q$. *[4]*

f By considering the factorized form of f(t) explain the shape of its graph. *[2]*

g A piano string oscillates when plucked. The displacement, x, from equilibrium as a function of time is modelled by:

$$\frac{d^2x}{dt^2} + 4x = 0$$

Show that a function of the form $x = f(t) = \cos(\omega t)$ solves this differential equation for a positive value of ω to be stated. *[4]*

h The piano string can be subjected to an external driving force from a tuning fork. The differential equation becomes:

$$\frac{d^2x}{dt^2} + 4x = \cos kt$$

Find a solution of the form $x = f(t) + g(k)\cos kt$ where $g(k)$ is a function to be found. *[3]*

i Resonance is a phenomenon in which the amplitude of the driven oscillation grows without limit. For what positive value of k will resonance occur? Justify your answer. *[2]*

2 *[Maximum mark: 30]*

This question is about modelling the long term numbers of a badger population.

The number of adults and juveniles in a badger population in year n is modelled by:

$$\begin{pmatrix} A_{n+1} \\ J_{n+1} \end{pmatrix} = \boldsymbol{M} \begin{pmatrix} A_n \\ J_n \end{pmatrix}$$

where $\boldsymbol{M} = \begin{pmatrix} 0.5 & 0.6 \\ 2 & 0.3 \end{pmatrix}$

a Draw a transition diagram to represent this model. Hence describe what each number in the model represents. *[6]*

The matrix $\boldsymbol{M}$ has eigenvalues λ_1 and λ_2 where $\lambda_1 > \lambda_2$. The corresponding eigenvectors are $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$.

b Find λ_1 and λ_2. *[4]*

c Find $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. *[6]*

In year 0, the initial state vector is $\boldsymbol{p}_0 = \begin{pmatrix} 100 \\ 20 \end{pmatrix}$

d If $\boldsymbol{p}_0 = \alpha\boldsymbol{v}_1 + \beta\boldsymbol{v}_2$, find appropriate values for α and β. *[3]*

e As n gets larger, find an approximate expression for $\begin{pmatrix} A_n \\ J_n \end{pmatrix}$. Hence find the long-term growth ratio of the population. *[6]*

f The badgers are considered a pest, so a change is made to the habitat, which affects the model so that the new transition matrix, $\boldsymbol{N}$, is $\begin{pmatrix} 0.5 & 0.6 \\ x & 0.3 \end{pmatrix}$. Find the upper bound on the value of x which will result in the long-term decline of the badger population. *[5]*

Mathematics: applications and interpretation
Higher level
Practice set B: Paper 1

Candidate session number

2 hours

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer **all** questions. Answers must be written within the answer boxes provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A copy of the mathematics: applications and interpretation formula booklet is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1 *[Maximum mark: 5]*

In this question, an *outlier* is defined as a piece of data which is more than two standard deviations above or below the mean.

The heights of eight children, in centimetres, are:

122 124 127 131 134 134 136 147

Determine whether any of the heights are outliers. *[5]*

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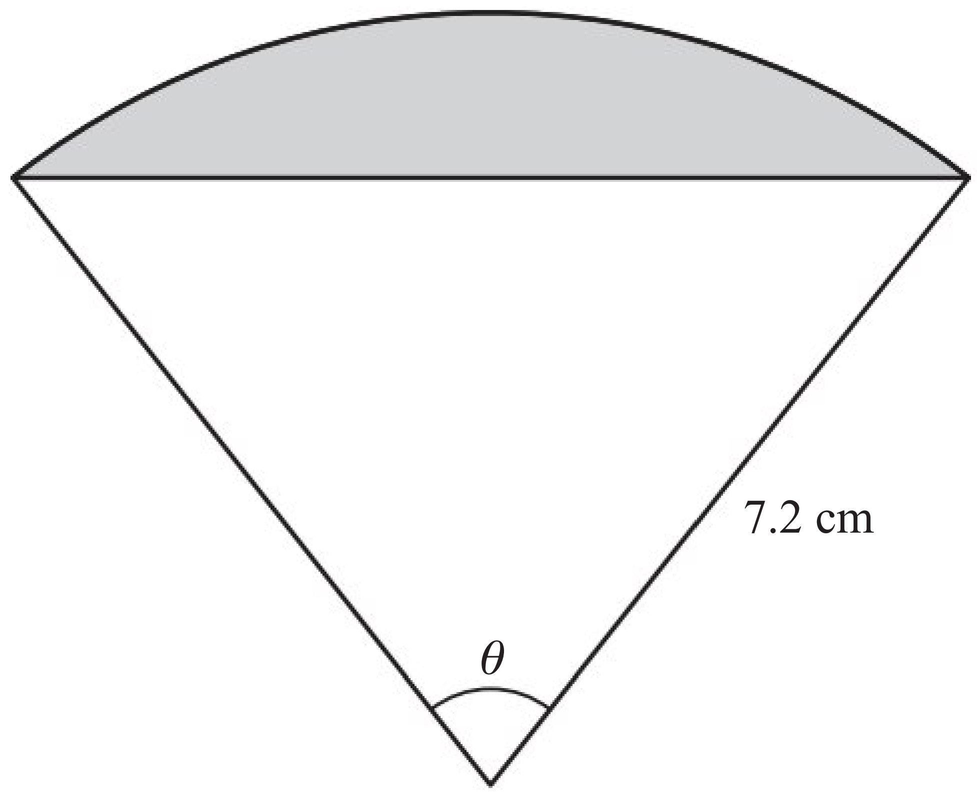
2 *[Maximum mark: 5]*

The diagram shows a sector of a circle with radius 7.2 cm. The angle at the centre is θ radians.



The area of the shaded area is $9.7\,\text{cm}^2$. Find the value of θ. *[5]*

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3 *[Maximum mark: 7]*

a Use the trapezoidal rule with five strips to estimate $\int_0^5 \sin\left(\frac{x^2}{10}\right) dx$, giving your answer correct to 3 s.f. *[4]*

b Use your GDC to evaluate the integral correct to 5 s.f. *[1]*

c Using the GDC value as the exact value, find the percentage error in the approximation obtained using the trapezoidal rule. *[2]*

4 *[Maximum mark: 6]*

A triangle ABC has sides $AB = 8$, $BC = 6$ and angle $BAC = \frac{\pi}{6}$ radians.

Find the two possible values of angle ABC. *[6]*

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5 *[Maximum mark: 5]*

On his way to school, Suresh stops for coffee with probability 0.8. If he stops for coffee, the probability that he is late for school is 0.4; otherwise, the probability that he is late is 0.1.

Given that on a particular day Suresh is late for school, what is the probability that he did not stop for coffee? *[5]*

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6 *[Maximum mark: 7]*

A business owner finds that the rate of change of profit from the sales of a particular product is given by $-40x^3 + 60x^2 + 30$, where x is the price in hundreds of dollars.

When the price is \$200 the profit is \$45.

a Find the profit function, P(x). *[5]*

b Find the price that maximizes the profit. *[2]*

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7 *[Maximum mark: 6]*

Given the functions

$f(x) = \frac{2 - x}{x + 3}$ $(x \neq -3)$ and $g(x) = \frac{2}{x - 1}$ $(x \neq 1)$,

find $(f \circ g)^{-1}$ in the form $\frac{ax + b}{cx + d}$. *[6]*

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8 *[Maximum mark: 6]*

A particle is moving in a straight line with velocity given by $v = 2e^{-t^2} - 1$ for $0 \le t \le 4$ where v is in $m\,s^{-1}$ and t is in s.

a Find the time at which the speed is zero. *[2]*

b Find the displacement from the initial position when $t = 4$. *[2]*

c Find the total distance travelled in the first four seconds. *[2]*

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9 *[Maximum mark: 7]*

The number of cars arriving at a multi-storey car park has a mean of 28 per hour. The cars arrive randomly.

a State two conditions that need to be satisfied to model the number of cars by a Poisson distribution. *[2]*

b Find the probability that more than 15 cars arrive in a 30-minute period. *[3]*

The number of motorbikes arriving at the car park is modelled by a Poisson distribution with mean 17 per hour.

c Find the probability that, in a given one-hour period, fewer than 40 vehicles (cars and motorbikes together) arrive at the car park. *[2]*

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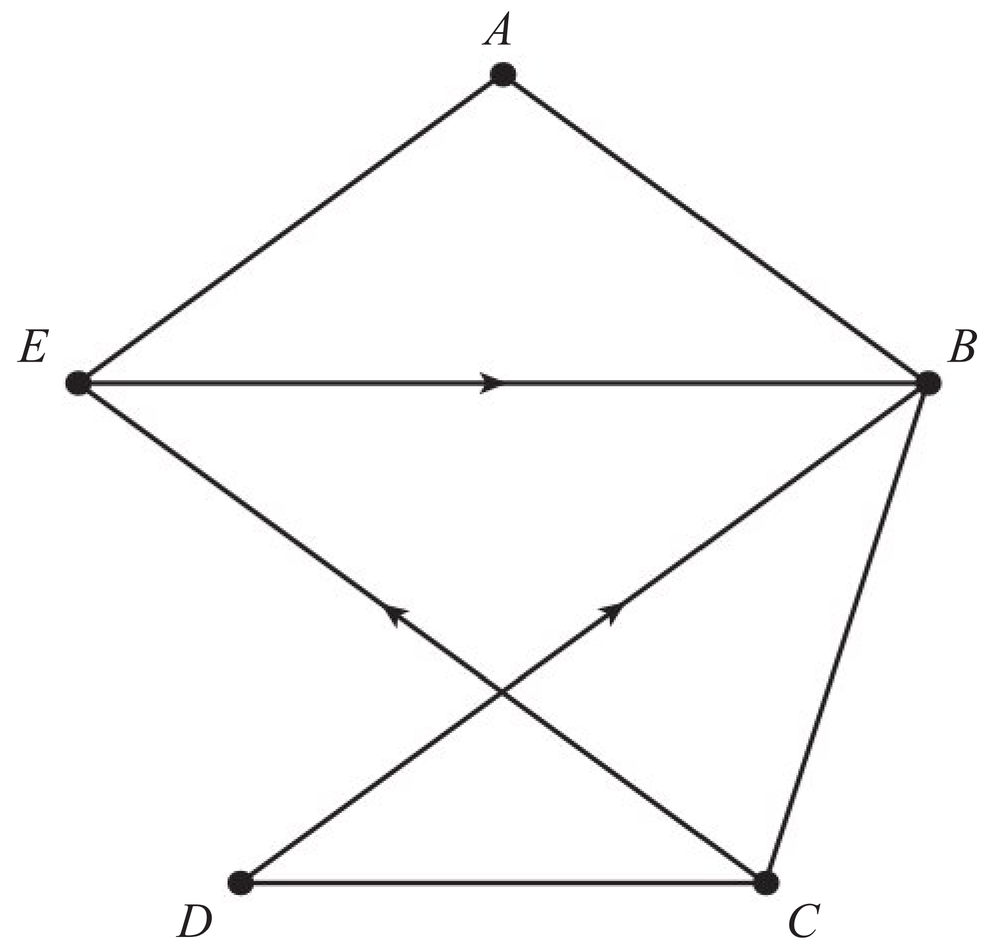
10 *[Maximum mark: 6]*

a Write down the transition matrix for the following directed graph: *[2]*



b The graph represents links between five web pages. For example, the edge A to B means that webpage A contains a link to webpage B. Use the PageRank algorithm to rank the pages in order of importance. *[4]*

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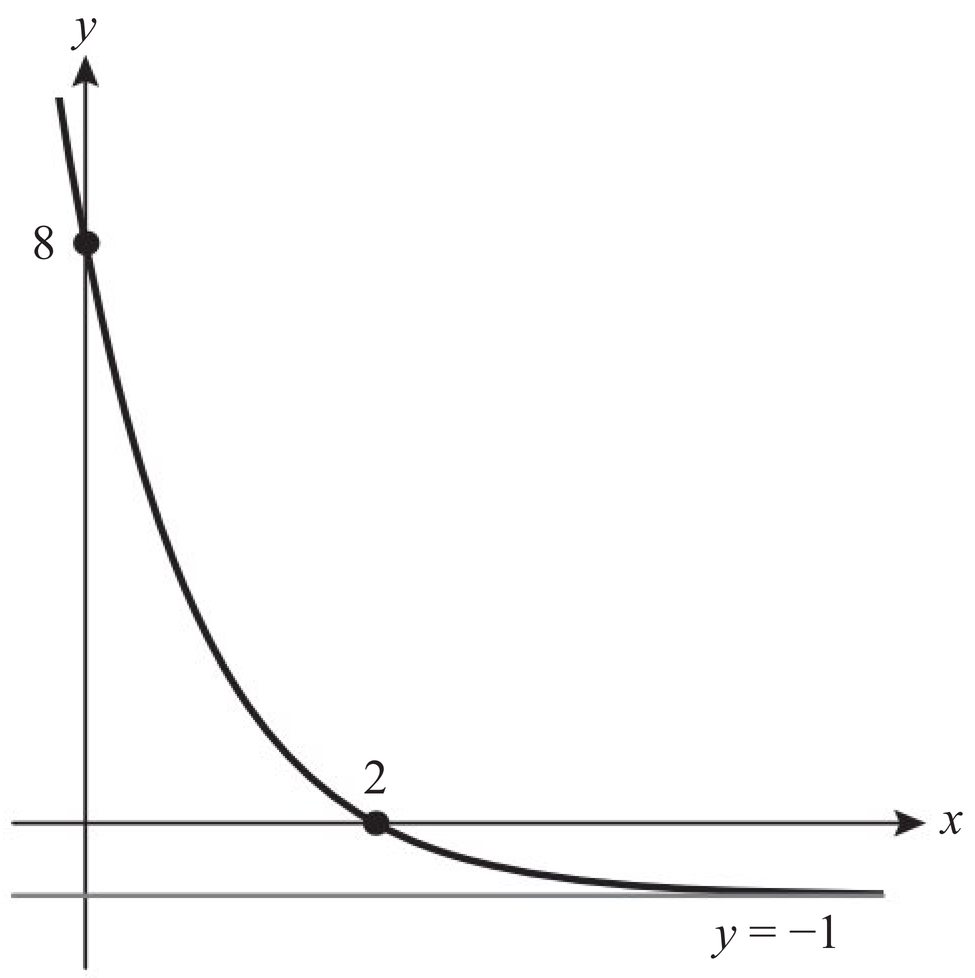
11 *[Maximum mark: 6]*

The graph in the diagram has equation $y = A + Be^{-kx}$.



Find the values of A, B and k. *[6]*

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12 *[Maximum mark: 6]*

A doctor is investigating whether a new drug decreases mean recovery time following a particular disease. It is known that, with the old drug, the recovery times follow a normal distribution with mean 12.6 days and standard deviation 2.8 days. The doctor wishes to conduct a hypothesis test using a 2% level of significance and a random sample of 40 patients. He assumes that the standard deviation is unchanged.

a Find the critical region for the test. *[3]*

b Given that the mean recovery time with the new drug is in fact 11.3 days, find the probability of a type II error. *[3]*

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13 *[Maximum mark: 7]*

Given the differential equation $\frac{dy}{dx} = \frac{x - y}{x + y + 1}$

a Sketch the slope field for $0 \le x \le 3$ and $0 \le y \le 3$. *[3]*

b Add the solution curve passing through (1, 1) to your diagram. *[1]*

c For the solution curve from part **b**, use Euler's method with step length 0.1 to estimate the value of y when $x = 1.5$. Give your answer to two decimal places. *[3]*

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14 *[Maximum mark: 9]*

The populations of spiders (x hundred) and flies (y hundred) are modelled by a system of differential equations.

In a simple model, the system has the form $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathbf{A}\begin{pmatrix} x-2 \\ y-3 \end{pmatrix}$. The equilibrium point of the system is (2, 3).

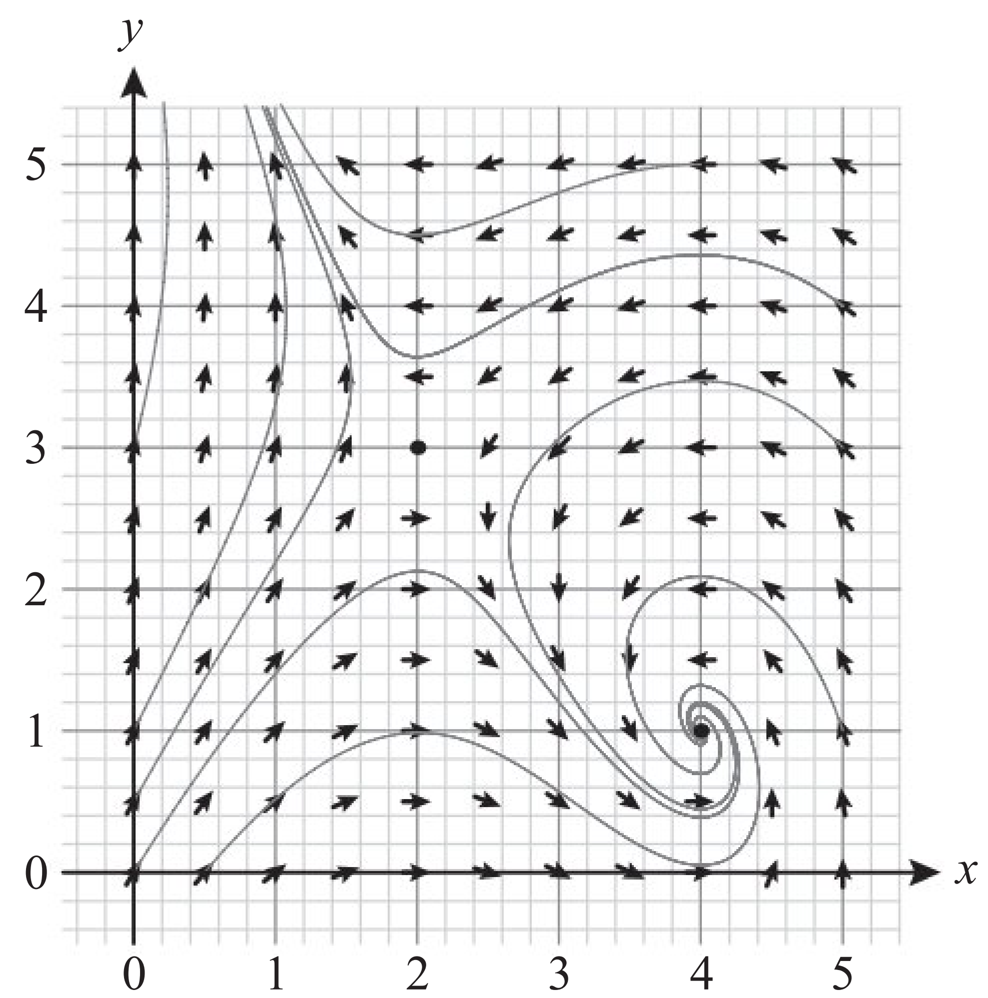
The matrix **A** has eigenvalues 0.6 and –0.4, and the corresponding eigenvectors $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

a Sketch the phase portrait for the system. *[2]*

b If initially there are 200 spiders and 300 flies, find the long-term relationship between the number of spiders and the number of flies, in the form $ax + by = c$. *[3]*

c Suggest why this model is not appropriate in the long term. *[1]*

In a refined model, there are two equilibrium points, (2, 3) and (4, 1). The phase portrait for the system is shown below. The horizontal axis shows the number of spiders and the vertical axis the number of flies.



d If initially there are 100 spiders and 200 flies, describe how the number of flies changes over time. *[1]*

e If initially there are 200 spiders and 100 flies, describe how the numbers of spiders and flies change in the long term. *[2]*

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15 *[Maximum mark: 6]*

Katie wants to buy a large bag of sugar. The weights of large bags of sugar follow a normal distribution with mean 2.01 kg and standard deviation 0.06 kg. The shop is out of large bags of sugar, so she buys two small bags instead. The weights of small bags of sugar follow a normal distribution with mean 1.08 kg and standard deviation 0.04 kg.

a Find the probability that a large bag of sugar weighs more than twice as much as a small bag. *[3]*

b Find the probability that a large bag of sugar weighs more than two small bags. *[3]*

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16 *[Maximum mark: 6]*

The height of a wave, h_1 cm, at time t seconds is given by the equation $h_1 = 12.3 \sin(3.2t)$. The height of another wave is given by $h_2 = 11.6 \sin(0.8 + 3.2t)$.

Find an expression for the height of the combined wave, $h = h_1 + h_2$, in the form $h = A \sin(k + bt)$. *[6]*

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17 *[Maximum mark: 10]*

The growth of a population of bacteria in a petri dish is modelled by the function

$$f(t) = \frac{L}{1 + Ce^{-kt}}$$

where $L, C, k > 0$.

a Show that $f'(t) = \dfrac{LCke^{-kt}}{(1 + Ce^{-kt})^2}$ *[2]*

The function has a point of inflection.

b Find, in terms of k and C, the time at which this occurs. *[6]*

c Find the range of possible values of C. *[2]*

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Mathematics: applications and interpretation
Higher level
Practice set B: Paper 2

Candidate session number

1 hour 30 minutes

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- You are permitted access to a graphical display calculator for this paper.
- Answer **all** questions in an answer booklet.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
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- The maximum mark for this examination paper is **[110 marks]**.

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1 *[Maximum mark: 20]*

Stella is planning to start a small business selling cosmetics gift boxes. She plans to start by selling 30 boxes in the first month. In each subsequent month she plans to sell 10 more boxes than in the previous month.

a **i** According to Stella's plan, how many boxes will she sell in the 12th month?

ii How many boxes will she sell in the first year?

iii In which month will she sell her 2000th box? *[8]*

Giulio also sells cosmetics gift boxes. He also sells 30 boxes in the first month, but expects to increase his sales by 10% each month.

b **i** How many boxes will Giulio sell in the first year?

ii In which month will Giulio first sell more than 1000 boxes per month? *[6]*

c Stella makes a profit of £2.20 per box and Guilio makes a profit of £3.10 per box.

i Find the profit each person makes in the first year.

ii In which month will Giulio's **total** profit first overtake Stella's? *[6]*

2 *[Maximum mark: 19]*

Laura is investigating whether a certain drug is effective in reducing cholesterol. She measures the cholesterol level (in mg dL^{-1}) of ten volunteers before and after a course of the drug:

Volunteer	A	B	C	D	E	F	G	H	I	J
Before	187	135	219	149	203	156	129	180	212	166
After	179	138	204	151	197	154	128	173	198	166

She carries out a paired test on the data at a 5% significance level.

a State two advantages of using a paired test over a two-sample test. *[2]*

b **i** Write down the hypotheses for the test.

ii Calculate the p-value for the test.

iii State the conclusion of the test. Give a reason for your answer. *[5]*

Laura decides she needs to assess the statistical validity of the test used in part **b**. She collects more data and summarizes the difference, d, between each participant's cholesterol level before and after the course of medication:

Difference	$-10 \le d < -5$	$-5 \le d < 0$	$0 \le d < 5$	$5 \le d < 10$	$10 \le d < 15$	$15 \le d < 20$
Observed frequency	3	14	30	24	17	5

c For these data find unbiased estimates of:

i the mean

ii the variance. *[2]*

Laura conducts a chi-squared goodness of fit test to determine whether these data are consistent with being from a normal distribution. The test is carried out at a 10% significance level.

d **i** Write down the hypotheses for this test.

ii Copy and complete the following table.

Difference	$d < -5$	$-5 \le d < 0$	$0 \le d < 5$	$5 \le d < 10$	$10 \le d < 15$	$d \ge 15$
Expected frequency						

iii Write down the number of degrees of freedom.

iv Find the p-value for the test.

v State the conclusion of the test. Give a reason for your answer. *[9]*

e Explain whether the result of this test supports the validity of the test in part **b**. *[1]*

3 *[Maximum mark: 15]*

The table shows the lengths (in km) of roads connecting seven villages. The information can also be represented on a weighted undirected graph.

	B	**C**	**D**	**E**	**F**	**G**
A	7	–	12	10	–	8
B	–	–	–	10	–	–
C		–	8	12	17	9
D			–	–	–	–
E				–	14	–
F					–	–

a Write down the degree of each vertex of the graph. *[2]*

b Draw the graph. *[2]*

c A road inspector would like to drive along each road exactly once and return to the starting point. Explain why it is possible to do this. Find the length of his route. *[3]*

d The road between A and G is closed. Find the length of the shortest route the inspector can take in order to drive along each road exactly once and return to the starting point. State which, if any, roads need to be used twice. *[4]*

e Internet cables are to be placed under the roads so that each village is connected, directly or indirectly, to village A. Use Kruskal's algorithm to find the minimum length of cable required. *[4]*

4 *[Maximum mark: 21]*

In a simple predator–prey model, the numbers of predators (x hundred) and prey (y thousand), at time t years, are modelled by the system of equations

$$\begin{cases} \dot{x} = -5x + 2y \\ \dot{y} = -6x + 3y \end{cases}$$

a Show that $\begin{pmatrix}1\\1\end{pmatrix}$ and $\begin{pmatrix}1\\3\end{pmatrix}$ are eigenvectors of the matrix $\begin{pmatrix}-5 & 2\\-6 & 3\end{pmatrix}$ and find the corresponding eigenvalues. *[3]*

b Sketch the phase portrait for the system, showing the directions of the eigenvectors and the direction of the trajectories. *[3]*

c Write down the general solution of the system. *[2]*

d In the case where initially there are 100 predators and 2000 prey, find expressions for x and y in terms of t, and then describe the long-term behaviour of the two populations. *[5]*

e In the case where initially there are 200 predators and 1000 prey, the prey population dies out when $t = T_0$.

i Find the value of T_0, and find the number of predators at that time.

ii After there are no more prey, the number of predators satisfies the equation $\dot{x} = -5x$. Find a new expression for the number of predators for $t > T_0$. *[8]*

5 *[Maximum mark: 18]*

A ball is thrown from the origin O with velocity $(8\mathbf{i} + 14\mathbf{j})\,\text{m s}^{-1}$. It moves freely under gravity so has acceleration $-9.8\mathbf{j}\,\text{m s}^{-2}$.

a Find the ball's velocity at time t. *[3]*

The ball passes through the point P t seconds after being thrown. The point Q is vertically below P on the same horizontal plane as O and $OQ = 2PQ$.

b Find the value of t. *[5]*

c Find the speed of the ball at P. *[2]*

The ball has the same speed at another point R as it does at P.

d Find the time taken for the ball to travel from O to R. *[2]*

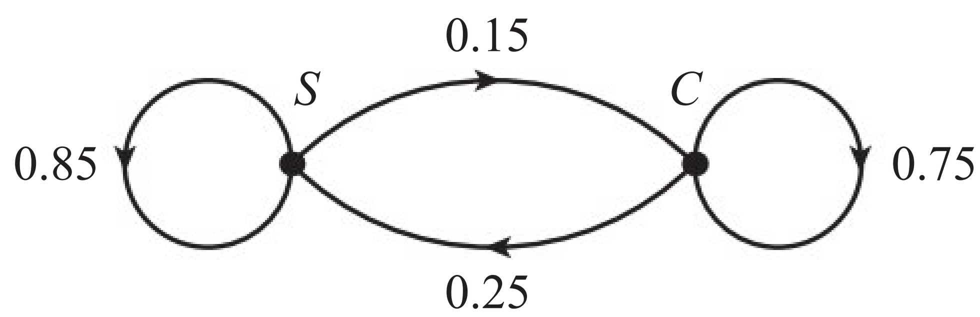
One second after the ball is thrown from O a second ball is thrown from O with velocity $(12\mathbf{i} + k\mathbf{j})\,\text{m s}^{-1}$.

e Given that the balls collide, find the value of k. *[6]*

6 *[Maximum mark: 17]*

There are two subscription television providers in the market, a satellite television company, S, and a cable television company, C.

The following graph shows the probabilities of customers changing between the two providers:



a Write down the transition matrix, **T**, for this graph. *[2]*

Initially, S has 9000 subscribers in a particular town and C has 7000.

b Find the number of subscribers each provider has after 4 years. *[2]*

c Find the eigenvalues and corresponding eigenvectors of **T**. *[4]*

d Hence write down matrices **P** and **D** such that $\mathbf{T} = \mathbf{PDP}^{-1}$. *[2]*

e Find an expression for the number of customers S has after n years. *[5]*

f Hence state the number of customers S can expect to have in the long term. *[1]*

g Give one reason why this model is unlikely to be accurate. *[1]*

Mathematics: applications and interpretation
Higher level
Practice set B: Paper 3

Candidate session number

1 hour

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- A graphical calculator is required for this paper.
- Answer **all** questions in an answer booklet.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A copy of the mathematics: applications and interpretation formula booklet is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1 *[Maximum mark: 25]*

This question is about estimating parameters from data.

Let X_1 and X_2 both be random variables representing independent observations from a population with mean μ and variance σ^2.

In this question you may use without proof the fact that

$\text{Var}(X) = \text{E}(X^2) - \text{E}(X)^2$

a Find an expression for $\overline{X}$, the random variable representing the sample mean of the two observed values. *[1]*

b Show that $\text{E}(\overline{X}) = \mu$ and find an expression for $\text{Var}(\overline{X})$ in terms of σ. *[4]*

The sample variance is defined as

$$S^2 = \frac{X_1^2 + X_2^2}{2} - \overline{X}^2$$

c **i** Find $\text{E}(X^2)$ in terms of $\text{Var}(X)$ and $\text{E}(X)$.

ii Show that $\text{E}(S^2) = \frac{1}{2}\sigma^2$. *[4]*

An unbiased estimator of a population parameter is one whose expected value equals the population parameter.

d **i** Show that $M = \frac{2X_1 + 3X_2}{5}$ is an unbiased estimator of μ.

ii When comparing two unbiased estimators, the one with a lower variance is said to be more efficient. Determine whether M or $\overline{X}$ is a more efficient unbiased estimator of μ. *[5]*

In a promotion, tokens are placed at random in boxes of cereal. Y is the random variable describing the number of boxes of cereal that need to be opened up to and including the one where a token is found. Two independent investigations were conducted.

e The tokens are placed in cereal boxes with probability p. The presence of a token in a cereal box is independent of other boxes.

i Find an expression for L, the probability of observing $Y_1 = a$ and $Y_2 = b$ in terms of a, b and p.

ii Find the value of p which maximizes L. This is called the maximum likelihood estimator of p. *[8]*

In the first observation, Y was found to be 4. In the second observation, Y was found to be 8.

f **i** Find an unbiased estimate for the variance of Y.

ii Find a maximum likelihood estimate for p. *[3]*

2 *[Maximum mark: 30]*

This question is about the path of three snails chasing after each other.

a Find $\left|e^{\frac{2i\pi}{3}} - 1\right|$. *[3]*

Three snails – Alf, Bill and Charlotte – are positioned on the vertices of an equilateral triangle whose centre of rotational symmetry is the origin of the Argand plane. Alf is positioned at the point $z = 1$ and Bill is above the real axis.

b Find the positions of the other two snails. *[2]*

c If Bill is stationary and Alf moves towards him at speed 1 unit per second, how far does Alf travel until he reaches Bill? How long does it take Alf to get there? *[2]*

Alf chases Bill, Bill chases Charlotte and Charlotte chases Alf. They all travel with speed 1 unit per second. The position of Alf at time t is denoted by z_A and the position of Bill is denoted by z_B.

d Explain why $\frac{dz_A}{dt} = \frac{z_B - z_A}{|z_B - z_A|}$ *[2]*

e Write z_B in terms of z_A. *[1]*

f If $z_A = re^{i\theta}$, find an expression for $\frac{dz_A}{dt}$ in terms of r, θ, $\frac{dr}{dt}$ and $\frac{d\theta}{dt}$. *[2]*

g Hence, by comparing real and imaginary parts, find differential equations for $\frac{dr}{dt}$ and $\frac{d\theta}{dt}$. *[7]*

h Solve the differential equations you found in part **g**. *[7]*

i How long does it take Alf to reach Bill? How far has Alf travelled until he reaches Bill? How many rotations does he make around the origin? *[4]*

Mathematics: applications and interpretation
Higher level
Practice set C: Paper 1

Candidate session number

2 hours

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer **all** questions. Answers must be written within the answer boxes provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A copy of the mathematics: applications and interpretation formula booklet is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1 *[Maximum mark: 5]*

In an arithmetic sequence, the fifth term is 7 and the tenth term is 81. Find the sum of the first 20 terms. *[5]*

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2 *[Maximum mark: 5]*

Sacha is investigating the relationship between time spent doing homework and time spent on social media. In her year at school, at the time of the survey, 60% of students are aged 17 and the rest are aged 18. Sacha wants to represent both age groups fairly, so she takes a random sample of six 17-year-olds and four 18-year-olds.

a State the name of this sampling technique. *[1]*

The results are shown in the table, showing the number of hours per day spent on each activity.

Student	1	2	3	4	5	6	7	8	9	10
Time spent on social media (x)	1.7	3.5	2.6	1.7	2.1	3.2	3.8	2.5	3.1	3.6
Time spent on homework (y)	4.2	2.1	3.2	3.5	4.2	2.5	0.6	2.5	2.7	1.5

Sacha finds that there is a strong negative correlation between the two variables, and decides to use a linear regression line to model the relationship between them.

b Find the equation of the regression line in the form $y = ax + b$. *[2]*

c Interpret, in the context of this question, the meaning of the coefficients a and b. *[2]*

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3 *[Maximum mark: 6]*

A child makes a caterpillar out of modelling clay. The density of the clay is $1.45\,\text{g}\,\text{cm}^{-3}$.

She starts by making a sphere of radius 3 cm.

a Find the mass of this sphere. *[2]*

She then adds more spheres, each with radius half the previous one.

b Show that the mass of the caterpillar can never reach 200 g. *[4]*

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4 *[Maximum mark: 6]*

Data collected over the course of recent seasons shows that the probability distribution of goals scored each game by Athletic Town is as follows:

Goals per game	0	1	2	3	4	>4
Probability	$\frac{1}{4}$	k	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	0

a Find the value of k. *[2]*

b Find the expected number of goals scored by Athletic Town in a 38 game season. *[4]*

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5 *[Maximum mark: 6]*

z is the complex number which satisfies the equation $3z - 4z^* = 18 + 21i$.

Find $\left|\frac{z}{3}\right|$. *[6]*

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6 *[Maximum mark: 8]*

Let $f(x) = \frac{1}{x+2}$ and $g(x) = \frac{1}{x}$

a Write down the transformation which maps the graph of $y = g(x)$ to $y = f(x)$. *[1]*

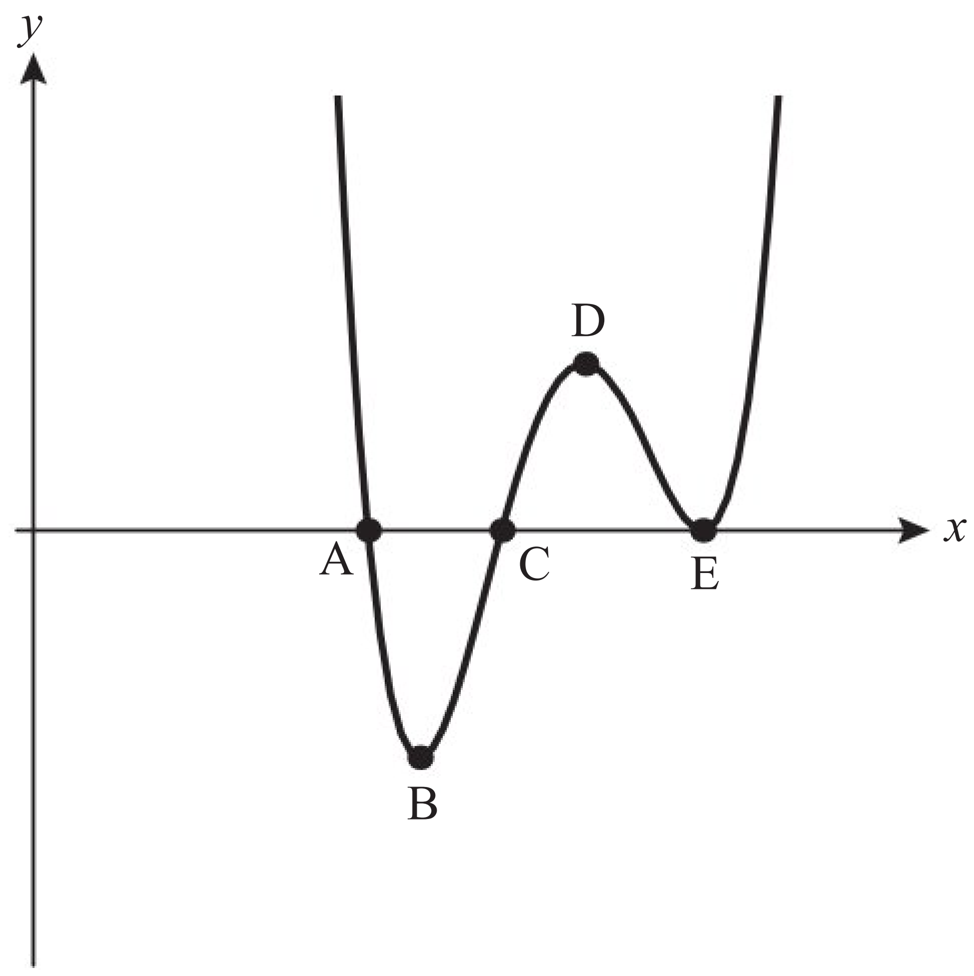
b Hence or otherwise write down the equations of the asymptotes of $y = f(x)$. *[2]*

c Find $f^{-1}(x)$. *[3]*

d Write down the equations of the asymptotes of $y = f^{-1}(x)$. *[2]*

7 *[Maximum mark: 5]*

The graph of $y = f'(x)$ is shown in the diagram.



Write down the labels of the following points, justifying your choice in each case:

a local maximum point(s) of $f(x)$ *[2]*

b point(s) of inflection of $f(x)$. *[3]*

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8 *[Maximum mark: 5]*

Vectors **a** and **b** satisfy

$\mathbf{a} \cdot \mathbf{b} = 17$ and $\mathbf{a} \times \mathbf{b} = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}$.

Find, in degrees, the size of the angle between the two vectors. *[5]*

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9 *[Maximum mark: 6]*

A particle moves in a straight line, with the velocity at time t seconds given by $v = \frac{\sin t}{\sqrt{t+1}}\,\text{m s}^{-1}$.

a Find the distance travelled by the particle in the first five seconds of motion, giving your answer to one decimal place. *[3]*

b Find the first two times when the magnitude of acceleration is $0.3\,\text{m s}^{-2}$. *[3]*

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10 *[Maximum mark: 6]*

Kishan is comparing the battery life of two brands of smartphone. He collects a random sample of eight phones of Brand A and ten phones of Brand B. The results, in hours, are as follows:

Brand A: 62, 56, 72, 81, 67, 66, 73, 78

Brand B: 83, 91, 76, 67, 73, 88, 75, 77, 81, 73

a State an assumption necessary in order to use a t-test. *[1]*

b Test, using a 5% significance level, whether there is evidence that the mean lifetimes for the two brands are different. State the hypotheses and conclusion clearly. *[5]*

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11 *[Maximum mark: 9]*

Two large rocks are submerged in the sea at points A and B with position vectors $(8\mathbf{i} - 3\mathbf{j})$ km and $(5\mathbf{i} + \mathbf{j})$ km, respectively.

A ship's captain wants to chart a course that keeps the ship equidistant from both rocks.

a Find the midpoint, M, of $\overrightarrow{AB}$. *[2]*

b Hence find the vector equation of the path the ship should follow. *[3]*

The ship is travelling at $20\,\text{km}\,\text{h}^{-1}$ and the ship starts on its course at 09:00.

c Given that the ship reaches M at 09:12, find the position vector of the ship at time t hours after 09:00. *[4]*

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12 *[Maximum mark: 5]*

In a large population of children, the times taken to read a page of text have mean 7.6 minutes and variance 3.7 minutes^2.

a Estimate the probability that the mean time for a sample of 40 children is more than 8 minutes. *[4]*

b Explain why it was necessary to use the Central Limit Theorem in your calculation in part **a**. *[1]*

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13 *[Maximum mark: 6]*

The force between two electrical charges is inversely proportional to the square of the distance between them: $F = \frac{k}{r^2}$, where F newtons is the force and r metres is the distance. When the two charges are 2 mm apart, the force between them is 3.6×10^{-6} newtons.

a Find the value of k, giving your answer in standard form. *[2]*

b The charges are moving apart from each other at a speed of $0.07\,\mathrm{m\,s^{-1}}$.
Find the rate at which the force between them is decreasing when they are 2 mm apart. *[4]*

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14 *[Maximum mark: 7]*

An ecologist wishes to develop a natural forest on a recently cultivated piece of land. There are three states: grass, g, shrubs, s, and trees, t, that the vegetation can be in. The forest is subdivided into a number of regions and each year these regions are observed and categorized as being in one of the three states.

From one year to the next, there is a 45% probability of a grass region becoming a shrubs region and a 5% chance of a grass region becoming a trees region.

There is a 10% chance of a shrubs region becoming a grass region and a 40% chance of it becoming a trees region.

There is a 0% chance of a trees region becoming a grass region and a 10% chance of it becoming a shrubs region.

a Write down the transition matrix for this system. *[2]*

The steady state of the system is G, S, T.

b Form a system of equations in G, S and T. *[3]*

c Hence find the exact steady state proportion of grass, shrubs and trees in the forest. *[2]*

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15 *[Maximum mark: 6]*

Find the particular solution of the differential equation

$$(3x^2 + 1)\frac{dy}{dx} = 4x$$

for which $y = 1$ when $x = 0$. *[6]*

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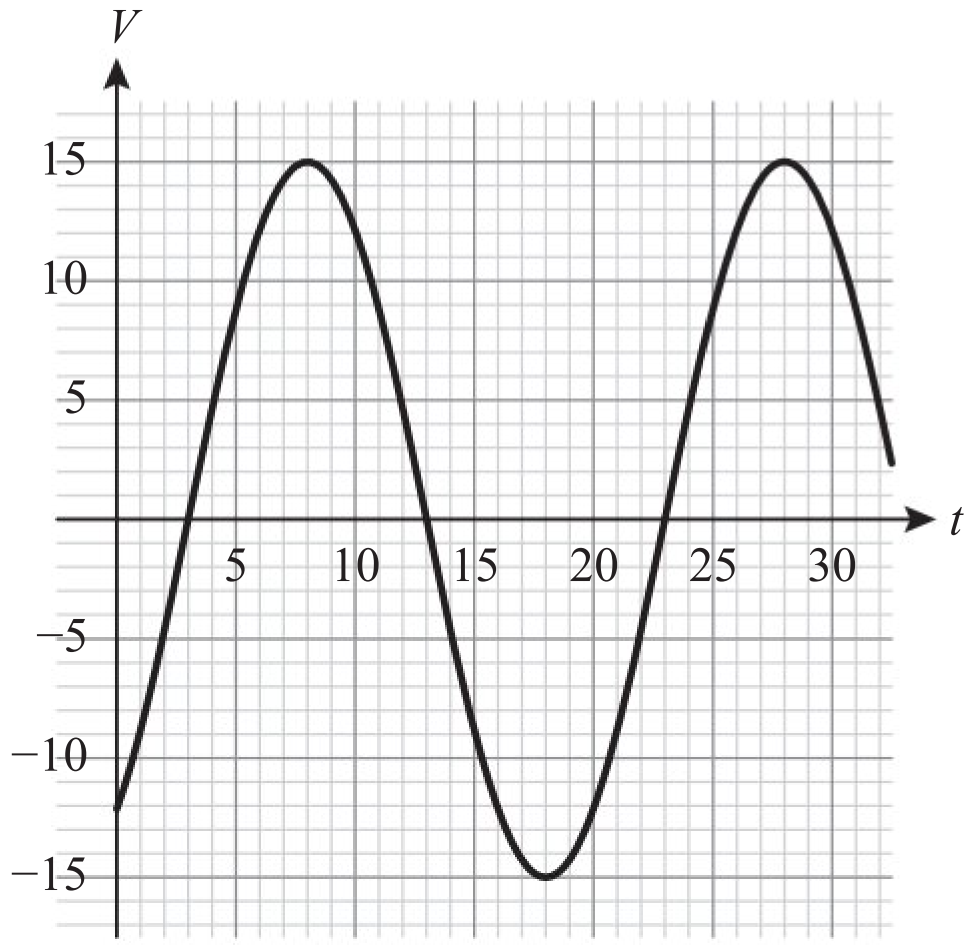
16 *[Maximum mark: 5]*

The voltage in an electric circuit is modelled by the function $V = a \sin(b(t - c))$. The graph of $y = V(t)$ is shown.



a Find the values of a, b and c. *[4]*

b On the diagram above, sketch the graph of $y = |V(t)|$. *[1]*

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17 *[Maximum mark: 8]*

The differential equation $\frac{d^2x}{dt^2} + 0.49x = 0$ is used to model the population of fish in a lake. The variable x is the number of fish above 400 at time t months.

a By writing $y = \frac{dx}{dt}$, express the equation as a system of two first order equations. *[2]*

b Find the eigenvalues of the corresponding matrix. Hence sketch the phase portrait for the system. *[4]*

c Describe how the number of fish in the lake changes over time. *[2]*

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18 *[Maximum mark: 6]*

The number of fish Bill catches per day at a particular lake follows a Poisson distribution with mean 8.7.

He thinks a new fishing rod will increase the number of fish he catches, so he conducts a hypothesis test by recording the number of fish caught the next day with his new rod.

a State the null and alternative hypotheses for this test. *[1]*

Bill wants the probability of making a Type I error to be less than 10%.

b Given that the mean number of fish caught with his new rod is actually 9.6, find the smallest possible probability of making a type II error. *[5]*

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Mathematics: applications and interpretation
Higher level
Practice set C: Paper 2

Candidate session number

1 hour 30 minutes

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- A graphical display calculator is needed for this paper.
- Answer **all** questions in an answer booklet.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A copy of the mathematics: applications and interpretation formula booklet is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1 *[Maximum mark: 21]*

The marks of Miss Rahman's class of 12 students on Mathematics Paper 1 and Paper 2 are given in the table.

Student	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
Paper 1	72	105	98	106	63	58	52	87	75	72	91	68
Paper 2	72	87	91	98	68	56	61	72	73	61	97	52

a Find the mean and standard deviation of each set of marks. Hence write two comments comparing the marks on the two papers. *[4]*

b The critical value of the Pearson's correlation coefficient for twelve pieces of data is 0.532. Determine whether there is significant positive correlation between the two sets of marks. *[3]*

c Two students did not sit Paper 2.

i Student 13 scored 95 marks on Paper 1. Use a regression line to estimate what mark he would have got on Paper 2.

ii Student 14 scored 45 marks on Paper 1. Can your regression line be used to estimate her mark for Paper 2? Justify your answer. *[5]*

d It is known that, in the population of all the students in the world who took Paper 1, the marks followed the distribution $N(68, 11^2)$. It is also known that 12% of all students achieved Grade 7 in this paper.

i How many of the 12 students in Miss Rahman's class achieved Grade 7 in Paper 1?

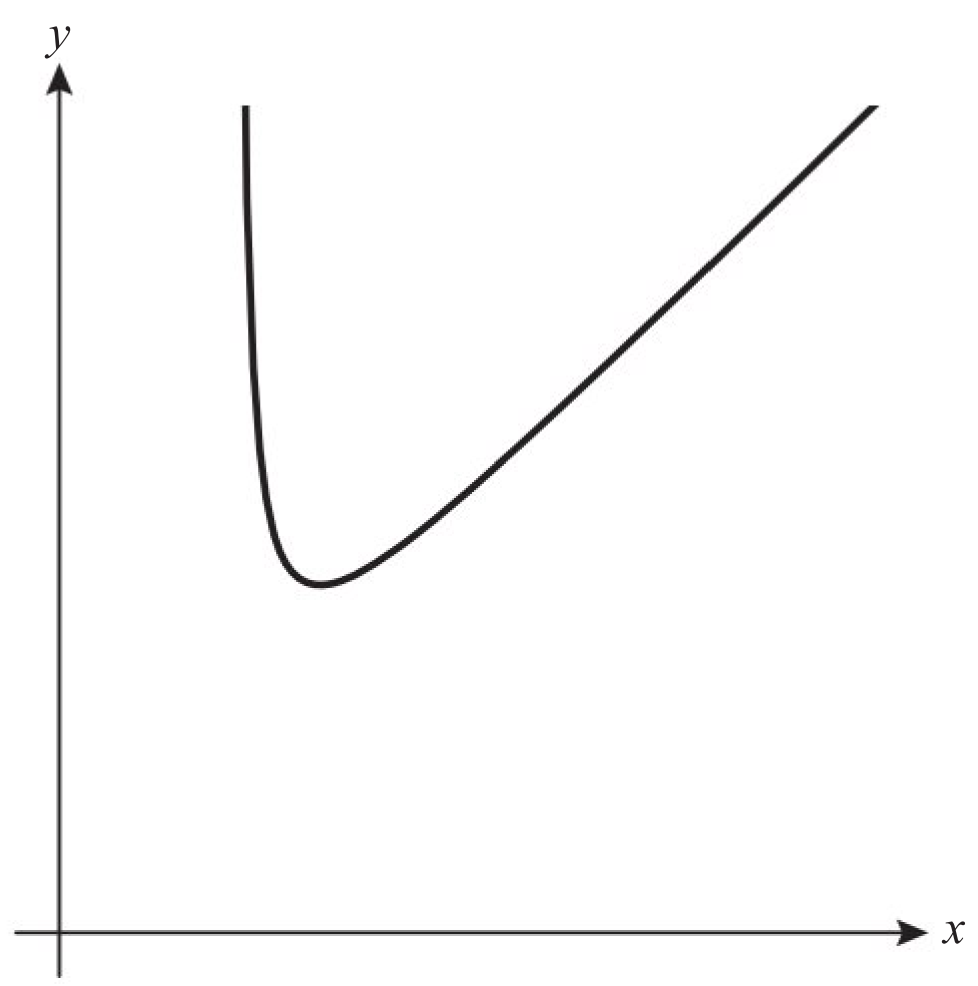
ii Find the probability that, in a randomly selected group of 12 students, there are more Grade 7s than in Miss Rahman's class. *[6]*

e Paper 1 is marked out of 110. To compare the results to another paper, Miss Rahman rescales the marks so that the maximum mark is 80.

Find the mean and standard deviation of the rescaled Paper 1 marks for the 12 students in the class. *[3]*

2 *[Maximum mark: 19]*

The graph below shows the function $f(x) = x + \frac{1}{x-a}$ for $x > a$, where $a > 0$.



a Find $f'(x)$. *[2]*

b The function $f(x)$ is increasing for $x > 3$. Find the value of a. *[3]*

c The normal to the graph is drawn at the point P where $x = 4$. The normal crosses the graph again at the point Q.

Find the coordinates of Q. *[5]*

d Find the area enclosed by the curve and the x-axis between the points P and Q. *[2]*

e Find the volume generated when the region from part **d** is rotated fully around the x-axis. *[3]*

f The value of a is now increased so that the area enclosed by the new curve, the x-axis and the lines $x = 4$ and $x = 5$ equals 5.
Find the new value of a. *[4]*

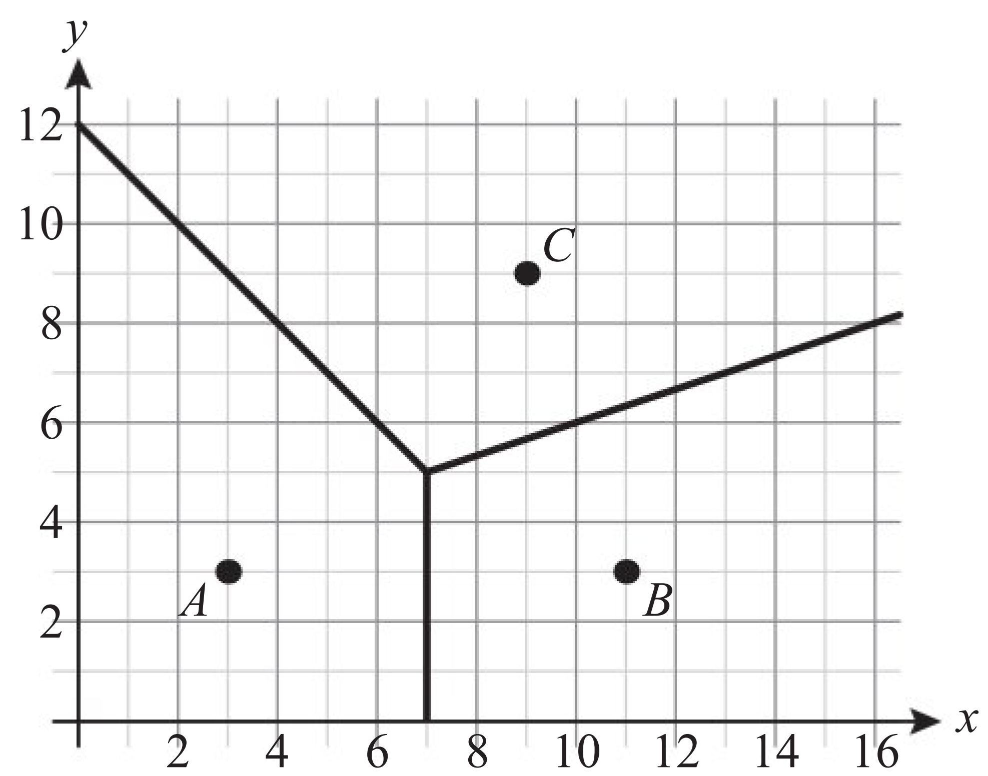
3 *[Maximum mark: 21]*

The diagram below shows a map of a forest, with one unit representing one kilometre. The boundary of the forest is determined by the lines with equations $x = 0$, $x = 16$, $y = 0$ and $y = 12$. Three log cabins are located at A(3, 3), B(11, 3) and C(9, 9).

a Find the distance from B to C. *[2]*

b Find the bearing of C from B. *[3]*

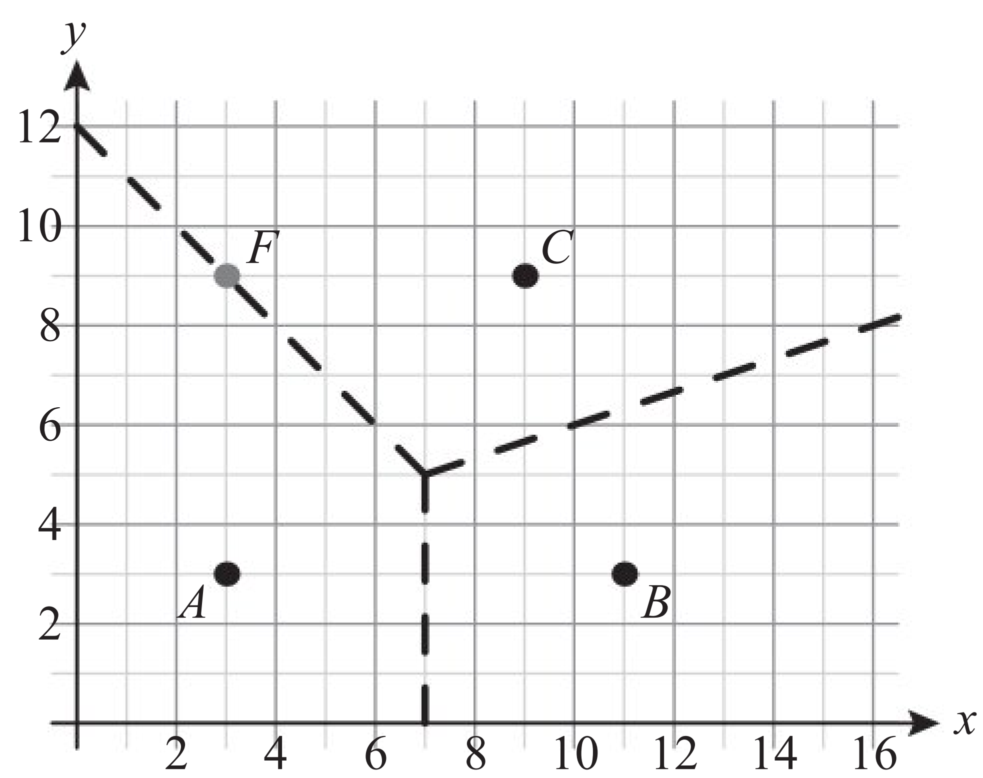
Each cabin is responsible for looking after a part of the forest. The areas of responsibility are the cells of the Voronoi diagram with sites A, B and C, as shown here.



c Write down the equation of the perpendicular bisector of AB and find the equation of the perpendicular bisector of BC.
Hence show that the vertex of the Voronoi diagram is located at the point with coordinates (7, 5). *[6]*

d Cabin A is responsible for an area of 59.5 km^2. Find the size of the areas of responsibility of cabins B and C. *[4]*

e Another cabin is built at the point F(3, 9). Draw the new Voronoi diagram, with sites A, B, C and F, on the diagram shown below. Show the edges of the new diagram as solid lines. (You do not need to show any equations of perpendicular bisectors, or calculations for the coordinates of the new vertex.) *[3]*



f A restaurant is to be built in the forest, inside the quadrilateral ABCF. The owners of the cabins want it to be as far as possible from any cabin. Find the coordinates of the location of the restaurant, showing all your working. *[3]*

4 *[Maximum mark: 17]*

The transformation T is represented by the matrix $\mathbf{M} = \begin{pmatrix} 5 & a \\ -1 & 7 \end{pmatrix}$

Find, in terms of a:

a the image of the point (–1, 2) under the transformation T. *[2]*

b the inverse matrix $\mathbf{M}^{-1}$. *[2]*

The transformation T maps the point P to the point with coordinates (a – 20, 11).

c Find the coordinates of P. *[3]*

For the rest of this question, $a = -3$.

d Find the eigenvalues and eigenvectors of **M**. *[4]*

e Hence write down the equation of the invariant lines of the transformation. *[2]*

The triangle S has vertices at (k, 2), (0, 3) and (0, 9), where k is a constant.

Triangle S is transformed to S' by the transformation T.

f Given that the area of S' is 720, find the value of k. *[4]*

5 *[Maximum mark: 16]*

The table shows the prices, in GBP, of flights between six cities available on a certain day.

		To					
		A	B	C	D	E	F
From	A	–	128	–	263	–	–
	B	–	–	–	112	–	–
	C	–	96	–	–	206	–
	D	312	–	–	–	108	–
	E	–	–	217	–	–	89
	F	–	–	–	–	72	–

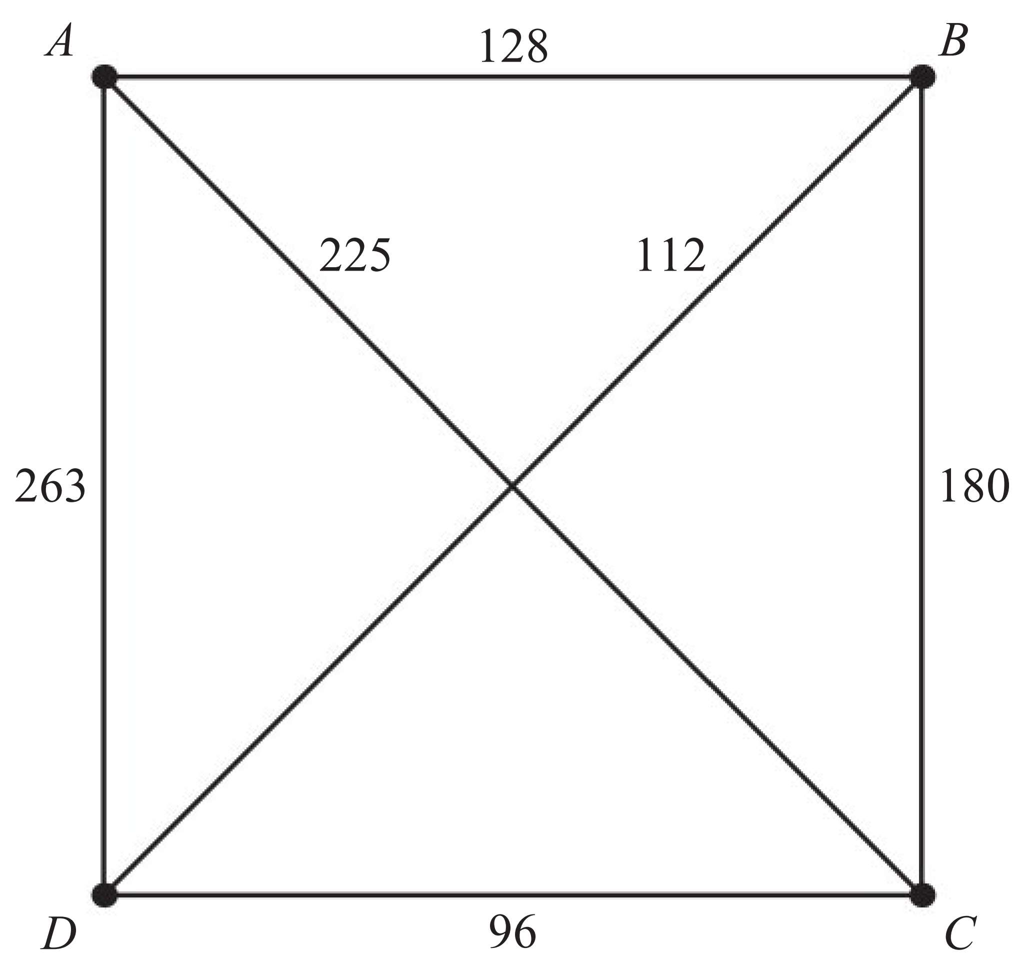
a What is the cheapest way to fly from A to D? State the route and the associated cost. *[2]*

b The flights are shown on a directed graph. Construct an adjacency matrix for this graph. *[2]*

c How many ways are there to fly from C to E using exactly three flights? *[2]*

d Jason is in city F. To which cities can he *not* fly using at most three flights? *[3]*

On a different day, the prices of direct flights between A, B, C, D are shown in the diagram below. The prices of flights between two cities are the same in both directions.



e Complete the table showing the cheapest way to fly (not necessarily directly) between each pair of cities. *[3]*

	B	C	D
A	128	225	
B	–	180	
C	–	–	

f Priya is a sales representative who needs to visit each of the four cities at least once and return to the starting point.

i Use the nearest neighbour algorithm to show that she can do this for a cost of at most 561 GBP.

ii By removing vertex A, find a lower bound for the cost of her trip.

iii Hence explain why the cheapest possible route for Priya costs exactly 561 GBP. *[4]*

6 *[Maximum mark: 16]*

The potential energy, V, between two helium atoms separated by a distance r is given by

$$V = \frac{A}{r^{12}} - \frac{B}{r^6}$$

where A, B are positive constants.

a Find the potential energy as the separation between the atoms becomes:

i very small

ii very large. *[2]*

b Find, in terms of A and B, the separation of the atoms when the potential energy is zero. *[2]*

The two particles are at their equilibrium separation when the potential energy between them is minimized.

c Find the equilibrium separation r_0. *[4]*

d **i** Find $\frac{d^2v}{dr^2}$

ii Hence justify that the value found in part **c** does give the minimum potential energy. *[5]*

The maximum binding energy of the atoms is the minimum potential energy.

e Show that the maximum binding energy is given by $-\frac{B^2}{2A}$. *[3]*

Mathematics: applications and interpretation
Higher level
Practice set C: Paper 3

Candidate session number

1 hour

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- A graphical display calculator is required for this paper.
- Answer **all** questions in an answer booklet.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A copy of the mathematics: applications and interpretation formula booklet is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1 *[Maximum mark: 25]*

This question is about an air traffic controller modelling the flight paths of various aircraft.

An air traffic controller has information about the trajectories of aircrafts relative to the base of his tower. The information is in terms of vectors:

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$$

where:

$x(t)$ is the displacement east in km

$y(t)$ is the displacement north in km

$z(t)$ is the vertical displacement from sea level in km

t is the time in hours after midnight.

a Helicopter A takes off at 9am from the point with position vector $\begin{pmatrix} 6500 \\ -4400 \\ 0 \end{pmatrix}$. It rises vertically at a rate of 3 km per hour.

i Find the position of helicopter A at time t hours after midnight for $t \geq 9$.

ii State one assumption of this model. *[3]*

b Plane B has trajectory $\begin{pmatrix} 600t \\ 1000 - 500t \\ 10 \end{pmatrix}$ for $t > 6$.

i Find the position of plane B at 9am.

ii Find the speed of plane B.

iii Show that A and B do not hit each other.

iv The air traffic controller must provide an alert if any two aircraft are on course to come within 10 km of each other. Determine whether the air traffic controller must provide an alert for helicopter A and plane B. *[9]*

c Just after take-off, the position of plane C is modelled by:

$$\begin{pmatrix} -100 + 100t + 5000t^2 \\ -200 + 200t + 10\,000t^2 \\ 10t + 500t^2 \end{pmatrix}$$

i Show that during this phase the plane is travelling in a straight line.

ii Find the angle of elevation of the plane's trajectory.

iii Find the magnitude of the acceleration of plane C during this phase. *[8]*

d Plane D is in a holding pattern with position modelled by

$$\boldsymbol{d} = \begin{pmatrix} 100 \sin 3\pi t \\ 100 \cos 3\pi t \\ t \end{pmatrix}$$

The vector **v** has components $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

i What is the geometric interpretation of the vector $\mathbf{q} = \mathbf{d} - (\mathbf{d} \cdot \mathbf{v})\mathbf{v}$?

ii The velocity of plane D is $\boldsymbol{v}_d$. Evaluate $\boldsymbol{v}_d \cdot \mathbf{q}$ and hence describe the trajectory of plane D. *[5]*

2 *[Maximum mark: 30]*

This question is about methods for studying the rate of a chemical reaction.

Juanita is a chemist, studying the rate of reaction of an enzyme catalyzed process. She expects a model of the form:

$$v = \frac{\alpha S}{\beta + S}$$

This is called the Michaelis–Menten model.

The dependent variable is v, the rate of the reaction. The independent variable is S, the concentration of the reactant.

There are two parameters of the system: α and β. Both are positive numbers.

a **i** Find $\frac{dv}{ds}$. Hence explain why v is an increasing function of S.

ii Find the limit of v as $S \to \infty$. Hence provide an interpretation for α.

iii By finding v when $S = \beta$ write down an interpretation of β. *[8]*

b Show that, if the Michaelis–Menten model is true, then the graph of $\frac{1}{v}$ against $\frac{1}{s}$ is expected to be a straight line.

Write down the gradient and the intercept of this line in terms of α and β. *[3]*

c Juanita makes five observations as shown in the table:

Observation	S	v
A	1	18
B	5	44
C	10	62
D	20	78
E	30	81

i Find the equation of the line of best fit of $\frac{1}{v}$ against $\frac{1}{s}$.

ii Hence estimate the values of α and β. *[5]*

d For the data from part **c**, a 5% significance test on the correlation coefficient is conducted.

i Find the value of the sample correlation coefficient for $\frac{1}{v}$ against $\frac{1}{s}$.

ii State the appropriate null and alternative hypotheses for the test.

iii State the p-value of the test, and hence the conclusion. *[5]*

e The instruments used to measure the rate of reaction have an error quoted as 10%. Find the percentage error in β if the true value of the rate of reaction is 10% higher than the value found in observation A. Comment on your answer. *[5]*

f Juanita reads in a book that a 95% confidence interval for the intercept in a regression model is

$$b_0 \pm 3.18\sqrt{MS_E\left(\frac{1}{n} + \frac{\bar{x}^2}{SS_x}\right)}$$

where:

b_0 is the value of the intercept found

$MS_E = \frac{\Sigma(y_i - \hat{y}_i)^2}{n-2}$

$\hat{y}_i$ is the value of y predicted by the regression line for the ith data item

$SS_x = \Sigma(x_i - \bar{x})^2$

Use Juanita's formula to find a 95% confidence interval for the value of α. *[4]*

Practice Set A: Paper 1 Mark scheme

1	**a**	$\frac{1}{4}$ or 10 seen	A1
		$\frac{10}{40} \times \frac{9}{39}$	(M1)
		$= \frac{3}{52}$	A1
			[3 marks]
	b	$\frac{10}{40} \times \frac{20}{39}$	(M1)
		$\times 2$	(M1)
		$= \frac{10}{39}$	A1
			[3 marks]
			Total [6 marks]
2	**a**	Find the diagonal of the square base: $\sqrt{8.3^2 + 8.3^2}$ [= 11.738]	M1
		Height $= \frac{11.738}{2}\tan(89.8°)$	M1
		$= 1681$	A1
		$= 1.7 \times 10^3$ cm	A1
			[4 marks]
	b	$\tan^{-1}\left(\frac{\text{height}}{\text{distance}}\right)$	
		Allow this mark even if the units are incorrectly converted.	M1
		$= 53.7°$	A1
			[2 marks]
			Total [6 marks]
3	**a**	H_0: The menu choice is independent of whether the person is a teacher or a student	
		H_1: The two factors are dependent	A1
		Attempt to calculate a chi-squared value or a p-value	M1
		$p = 0.0104$ or $\chi^2 = 9.12$	A1
		Comparison: $p < 0.05$	M1
		There is sufficient evidence that the menu choice is dependent on whether the person is a teacher or a student.	A1
			[5 marks]
	b	e.g. She could choose another day with a different menu.	A1
			[1 mark]
			Total [6 marks]
4	**a**	Use x and y values to set up three equations	M1
		$a + b + c = 7.2$	
		$4a + 2b + c = 7.4$	
		$9a + 3b + c = 6.4$	A1
		$a = -0.60$, $b = 2.0$, $c = 5.8$	A1
			[3 marks]
	b	5.8 m	A1**ft**
			[1 mark]
	c	Use GDC to solve $-0.60x^2 + 2.0x + 5.8 = 0$	M1
		$x = 5.2$ (m)	A1
			[2 marks]
	d	$\sqrt{5.2^2 + 5.8^2}$	(M1)
		$= 7.8$ m	A1
			[2 marks]
			Total [8 marks]
5	**a**	Limits $\sqrt[3]{5}$, $\sqrt[3]{17}$ (seen in either part)	A1
		$x = \sqrt{y^3 - 1}$	A1
		$\int \sqrt{y^3 - 1}\,dy$	M1
		$= 2.57$	A1
			[4 marks]
	b	Using x^2	M1
		$\int \pi (y^3 - 1)\,dy$	M1
		$= 24.9$	A1
			[3 marks]
			Total [7 marks]

6 **a** Using z-interval M1
$\bar{x} = 800.7$ (M1)
[796, 805] A1
[3 marks]

b z-interval because the underlying population distribution is normal with a known standard deviation. A1
[1 mark]

c 790 is outside the confidence interval M1
So the customer's claim does not seem justified. A1
[2 marks]
Total [6 marks]

7 **a** $C = 10t + 85$ (M1)(A1)
Note: Award M1 for linear model with correct y-intercept.
[2 marks]

b For $0 < t < 2$: $C = 20t + 60$ A1
For $t \geq 2$: $C = 30t + C$ (M1)
$20(2) + 60 = 60(2) + C$ M1
$C = -20$
$C = 60t - 20$ A1
Or can be given as $C = \begin{cases} 20t + 60, & 0 < t < 2 \\ 60t - 20, & t \geq 2 \end{cases}$
[4 marks]

c $10t + 85 = 20t + 60$ M1
$t = 2.5$, which is not in domain $0 < t < 2$
$10t + 85 = 60t - 20$ M1
$t = 2.1$
So minimum hire time is 2.1 hours A1
[3 marks]
Total [9 marks]

8 **a** Solve $0.003x^3 + 10x + 200 = 720$ using GDC M1
36 cakes A1
[2 marks]

b Sketch graph of $y = \frac{T(x)}{x}$ M1

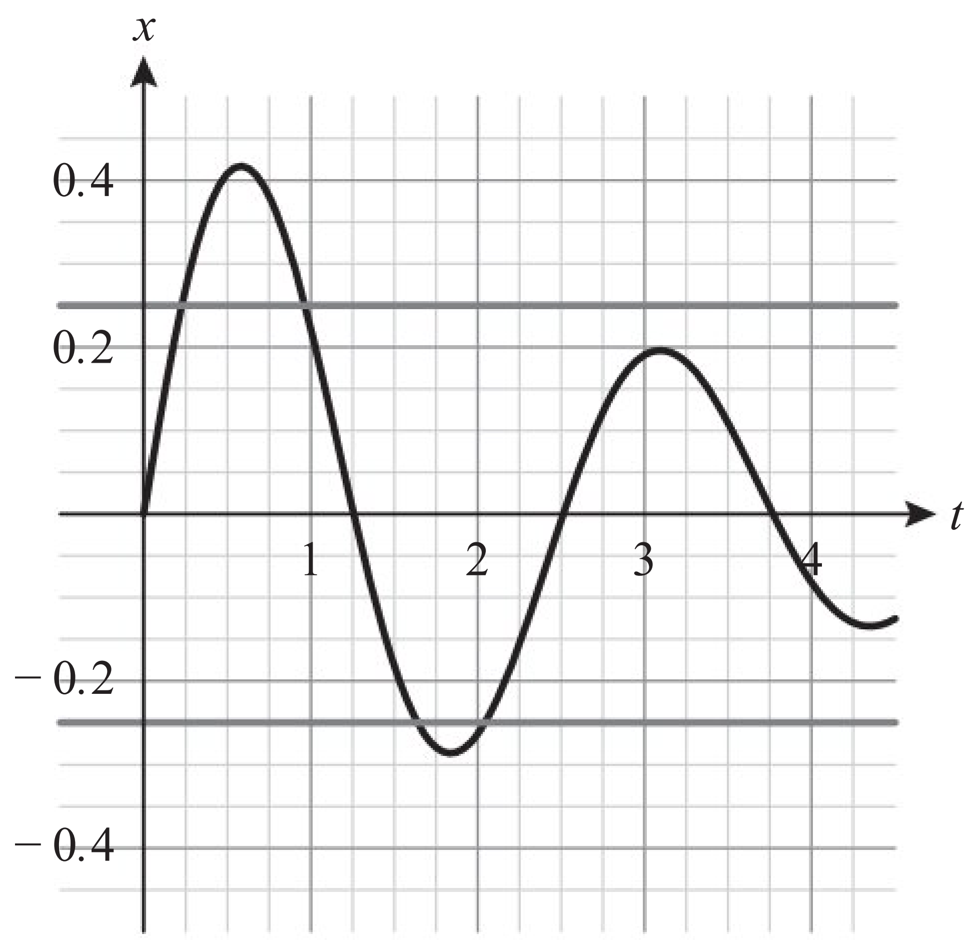
Minimum point marked at $x = 32.2$ M1
Min = 19.3 minutes A1
Max = 21.2 minutes A1
[4 marks]

c $S(x) = T\left(\frac{1}{2}x\right)$ M1
$= 0.003\left(\frac{1}{2}x\right)^3 + 10\left(\frac{1}{2}x\right) + 200$
$= 0.000375x^3 + 5x + 200$ A1
[2 marks]
Total [8 marks]

9 Use $\frac{u_1}{(1-r)} = 5$ M1

Use $u_1 + u_1 r = 3$ M1
Express u_1 from both equations and equate:
$5(1 - r) = \frac{3}{(1+r)}$ M1
$1 - r^2 = \frac{3}{5}$ A1
$r = \sqrt{\frac{2}{5}}$ A1
Total [5 marks]

10 a Use GDC to draw the graph with lines at $x = 0.25$ and $x = -0.25$ A1



Award M1A0 if only $x = 0.25$ considered. M1

4 times. A1

[3 marks]

b Attempt to use the product rule. M1

$-0.15e^{-0.3t}$ or $1.25 \cos(2.5t)$ seen M1

$\frac{dx}{dt} = -0.15e^{-0.3t} \sin(2.5t) + 1.25e^{-0.3t} \cos(2.5t)$ A1

[3 marks]

Total [6 marks]

11 The shortest edge from A is AB, so add B next. M1

The shortest edge from A or B is AC, so add C next M1

The next shortest edge is BC, but both B and C have already been added, so skip. A1

The shortest edge between a selected and an unselected vertex is BF, so add F next. These are the edges selected so far: M1

	A	B	C	D	E	F
A	–	11	12	25	–	–
B	**11**	–	14	–	22	18
C	**12**	14	–	–	24	12
D	25	–	–	–	31	31
E	–	22	24	12	–	35
F	–	**18**	20	31	35	–

Next add E (BE = 22) and finally D (ED = 12) A1

The weight of the tree is (11 + 23 + 18 + 22 + 12) = 86 A1

Total [6 marks]

12 a $\int \frac{1}{C^2} dC = \int -k \, dt$ M1

$-\frac{1}{C} = -kt + c$ A1A1

Note: award A1 for LHS and A1 for RHS

Substitutes in : $t = 0$, $C = C_0$: M1

$-\frac{1}{C_0} = c$ A1

$\frac{1}{C} = kt + \frac{1}{C_0}$ M1

$\frac{C_0}{C} = C_0 kt + 1$ AG

$C = \frac{C_0}{C_0 kt + 1}$

[6 marks]

	b	As $t \to \infty$, $\dfrac{C_0}{C_0kt + 1} \to 0$	M1
		So long-term concentration is zero.	A1
			[2 marks]
			Total [8 marks]
13	**a**	$3 + 3i$	A1
			[1 mark]
	b	Multiply by $re^{i\theta}$ where $r = 2$	M1
		and $\theta = \dfrac{\pi}{3}$	A1
		$b = -2.20 + 8.20i$ or $b = (3 - 3\sqrt{3}) + (3 + 3\sqrt{3}\ i)$	(A1)
		So the coordinates are (–2.20, 8.20)	A1
			[4 marks]
			Total [5 marks]
14	**a**	e.g. The sample would exclude households where everyone is at work or school.	A1
			[1 mark]
	b	H_0: $p = \dfrac{1}{5}$, H_1: $p > \dfrac{1}{5}$	A1
		Using $X \sim B\left(70, \dfrac{1}{5}\right)$	M1
		Probability calculations shown (looking for $P(X \geq k) < 0.05$ or $P(X \leq k - 1) > 0.95$)	M1
		$P(X \leq 19) = 0.945 < 0.95$	
		$P(X \leq 20) = 0.970 > 0.95$	A1
		The critical region is $X \geq 21$	A1
			[5 marks]
			Total [6 marks]
15	**a**	gradient $= \dfrac{-0.8}{6} \left(= -\dfrac{4}{30}\right)$	(M1)
		intercept = 4.6	(M1)
		$\ln m = 4.6 - \dfrac{4}{30}t$	A1
			[3 marks]
	b	$m = e^{4.6 - \frac{4}{30}t}$	M1
		$= 99.5e^{-\frac{4}{30}t}$	A1
			[2 marks]
			Total [5 marks]
16	**a**	$\|\mathbf{r}\| = \sqrt{a^2\cos^2 kt + a^2\sin^2 kt}$	M1
		$= \sqrt{a^2(\cos^2 kt + \sin^2 kt)}$	
		$= a$	A1
		Object is at a fixed distance a from the origin so is moving in a circle	R1
			[3 marks]
	b	$\mathbf{v} = \begin{pmatrix} -ka\sin kt \\ ka\cos kt \end{pmatrix}$	M1A1
		Note: Award M1 for attempt to differentiate	
		$\mathbf{r} \cdot \mathbf{v} = (a\cos kt)(-ka\sin kt) + (a\sin kt)(ka\cos kt)$	M1
		$= 0$	A1
		So the vectors **r** and **v** are perpendicular	
			[4 marks]
			Total [7 marks]
17		$\dfrac{dS}{dt} = 2\pi r\dfrac{dr}{dt}$ ...	A1
		... $+ \pi\dfrac{dr}{dt}\sqrt{r^2 + 25}$...	A1
		... $+ \pi r\dfrac{2r\frac{dr}{dt}}{2\sqrt{r^2 + 25}}$	M1A1
		Substitute $r = 10$, $\dfrac{dr}{dt} = 2$ into their expression	M1
		$\dfrac{dS}{dt} = 252\,\text{cm}^2\,\text{s}^{-1}$	A1
			Total [6 marks]

Practice Set A Paper 2: Mark scheme

1 a i $N = 25$
$I\% = 2$
$PV = -135\,000$
$FV = 0$
$P/Y = 1$
$C/Y = 1$ (M1)(A1)
Note: Award M1 for an attempt to use financial package on GDC; award A1 for all entries correct.
Payment per year = £6914.76 A1

ii 6914.76×25 (M1)
= £172 869 A1

iii $172\,869 - 150\,000$ (M1)
= £22 869 A1

[7 marks]

b i $N = 360$
$I\% = 2.5$
$PV = -150\,000$
$FV = 0$
$P/Y = 12$
$C/Y = 12$ (M1)(A1)
Note: Award M1 for an attempt to use financial package on GDC; award A1 for all entries correct.
Payment per month = £592.68 A1

ii 592.68×360 (M1)
= £213 364.80 A1

[5 marks]

c EITHER
Suresh should choose Mortgage A... A1
...because total repayment is lower than Mortgage B R1
OR
Suresh should choose Mortgage B... A1
...because there is no deposit/he could invest the £15 000 R1
Note: Do not award R0A1.

[2 marks]

d $N = 360$
$I\% = 0$
$PV = -15\,000$
$FV = 0$
$P/Y = 12$
$C/Y = 12$ (M1)(A1)
Note: Award M1 for an attempt to use financial package on GDC; award A1 for all entries correct.
Interest rate > 1.25% A1

[3 marks]

e $250 \times 1.02 + 250 \times 1.02^2 + \cdots + 250 \times 1.02^n$ M1A1
$= 250 \times 1.02 \left(\frac{1 - 1.02^n}{1 - 1.02}\right)$ M1
$= 12\,750(1.02^n - 1)$ A1

[4 marks]

Total [21 marks]

2 a $\cos\theta = \frac{2^2 + 4^2 - 4^2}{2 \times 2 \times 4}$ [= 0.25] (M1)
$\sin\theta = \sqrt{\frac{15}{16}}$ [= 0.968] (M1)
Area $= \frac{1}{2}(2 \times 4) \times$ their $\sin\theta$ M1
= 3.87 [cm^2] A1

[4 marks]

b The third side is $10 - 3x$ … M1
… which must be positive A1

[2 marks]

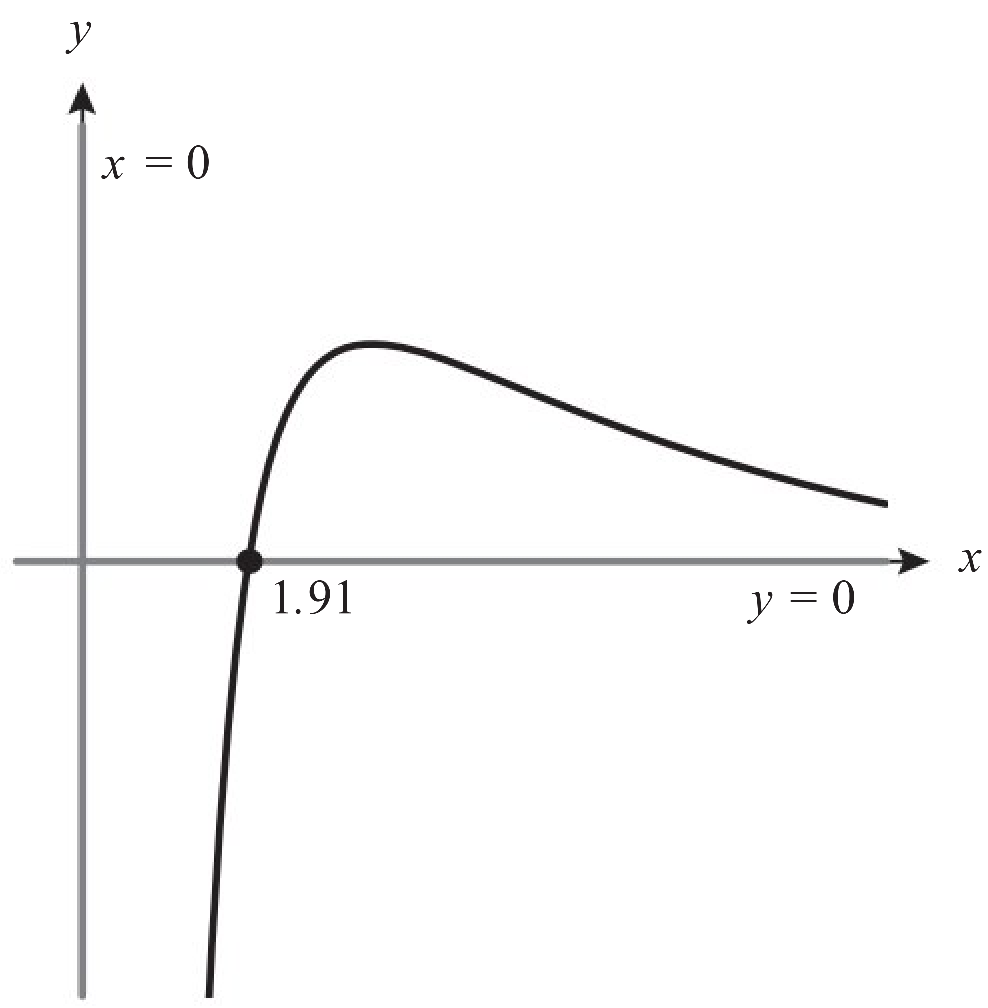
c **i** $(10 - 3x)^2 = x^2 + (2x)^2 - 2x(2x)\cos\theta$ M1

$100 - 60x + 9x^2 = 5x^2 - 4x^2\cos\theta$ A1

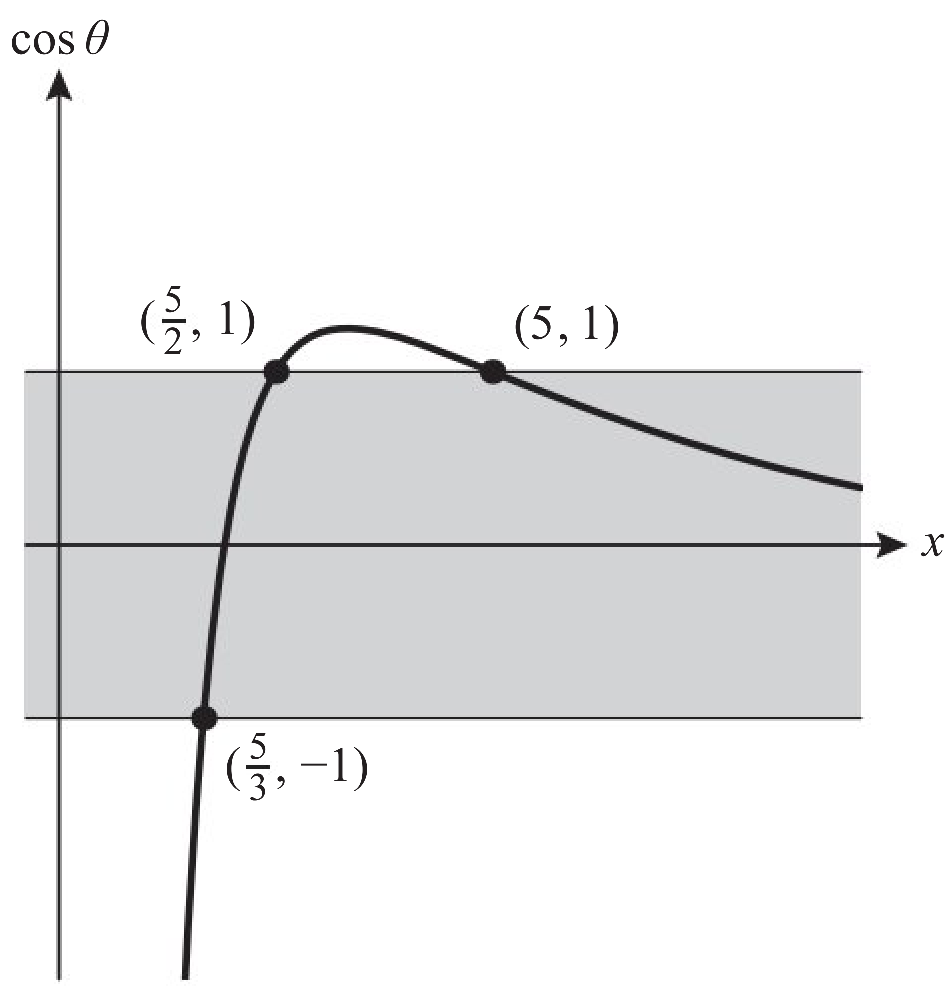
$\cos\theta = \dfrac{60x - 4x^2 - 100}{4x^2}$ M1

$= \dfrac{15x - x^2 - 25}{x^2}$ (AG)

ii A2



iii Need $-1 < \cos\theta < 1$ (allow ≤ here) M1



Intersections at $x = \frac{5}{3}, \frac{5}{2}, 5$ A1

So $\frac{5}{3} < x < \frac{5}{2}$ A1

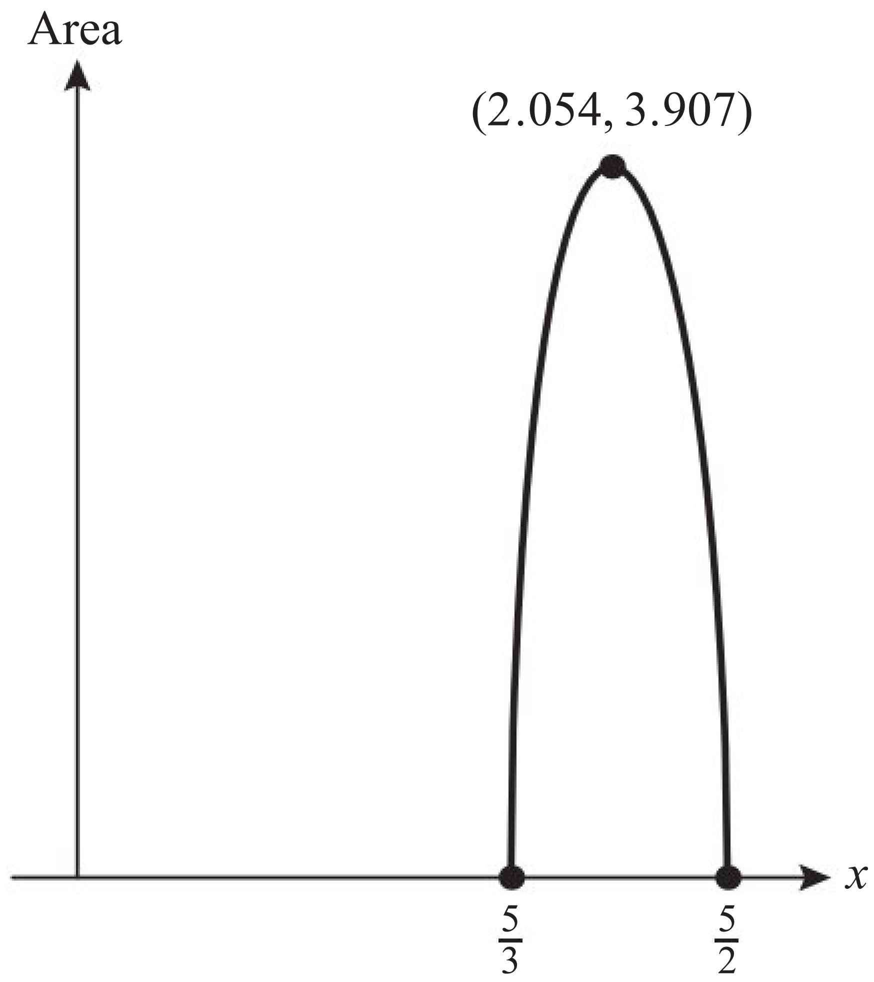
[8 marks]

d State or use $\sin\theta = \sqrt{1 - \cos^2\theta}$ M1

State or use Area $= \frac{1}{2}x(2x)\sin\theta$ M1

Use $\cos\theta = \dfrac{15x - x^2 - 25}{x^2}$ M1

Sketch area as a function of x: M1



Max area for $x = 2.05$ A1

Max area = 3.91 [cm^2] A1

[6 marks]

Total [20 marks]

3 a speed $= \sqrt{(-3)^2 + 7^2 + 2^2}$ M1

$= 7.87\,\text{km}\,\text{h}^{-1}$ A1

[2 marks]

b $\mathbf{r}_D = \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix} + t\begin{pmatrix} -3 \\ 7 \\ 2 \end{pmatrix}$ A1

$\mathbf{r}_E = \begin{pmatrix} -2 \\ 1 \\ -8 \end{pmatrix} + t\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ A1

[2 marks]

c i $\begin{cases} 4 - 3\lambda = -2 + 2\mu \ (1) \\ -2 + 7\lambda = 1 - \mu \ (2) \\ 1 + 2\lambda = -8 + 3\mu \ (3) \end{cases}$ M1

Solve any two simultaneously: $\lambda = 0$, $\mu = 3$ M1A1

Check these values in the third equation M1

So the paths cross.

ii (4, −2, 1) A1

[5 marks]

d D is at (4, −2, 1) when $t = 0$ but E is at (4, −2, 1) when $t = 3$ so they don't collide R1

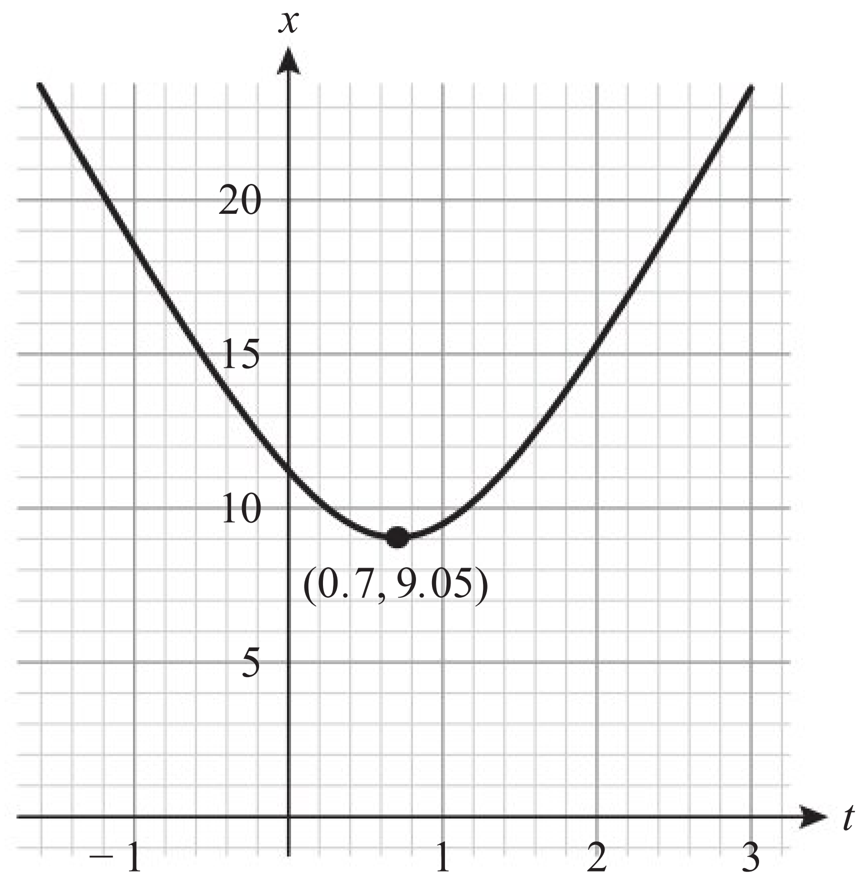
[1 mark]

e i $\overrightarrow{DE} = \begin{pmatrix} -2 + 2t \\ 1 - t \\ -8 + 3t \end{pmatrix} - \begin{pmatrix} 4 - 3t \\ -2 + 7t \\ 1 + 2t \end{pmatrix}$ M1

$= \begin{pmatrix} -6 + 5t \\ 3 - 8t \\ -9 + t \end{pmatrix}$ A1

$|\overrightarrow{DE}| = \sqrt{(-6 + 5t)^2 + (3 - 8t)^2 + (-9 + t)^2}$ M1

Sketch distance as a function of t: M1



$t = 0.7$ hours A1

ii $d_{\text{min}} = 9.05\,\text{km}$ A1

[6 marks]

Total [16 marks]

4 a As $t \to \infty$, $P \to \dfrac{5}{1+0} = 5$ (M1)

So long term population is 5000 A1

[2 marks]

b $\dfrac{5}{P} - 1 = Ce^{-kt}$ A1

$\ln\left(\dfrac{5}{P} - 1\right) = \ln Ce^{-kt}$ M1

$\ln\left(\dfrac{5}{P} - 1\right) = \ln C + \ln e^{-kt}$ M1

$\ln\left(\dfrac{5}{P} - 1\right) = \ln C - kt$ AG

[3 marks]

c $\ln\left(\dfrac{5}{P} - 1\right) = -0.251t + 0.973$ M1A1

$k = 0.251$ A1

$C = e^{0.973}$ M1

$C = 2.65$ A1

[5 marks]

d i $R^2 = 0.996$ A1

ii Since 0.996 is very close to 1... R1

...this suggests a very good fit for the model A1

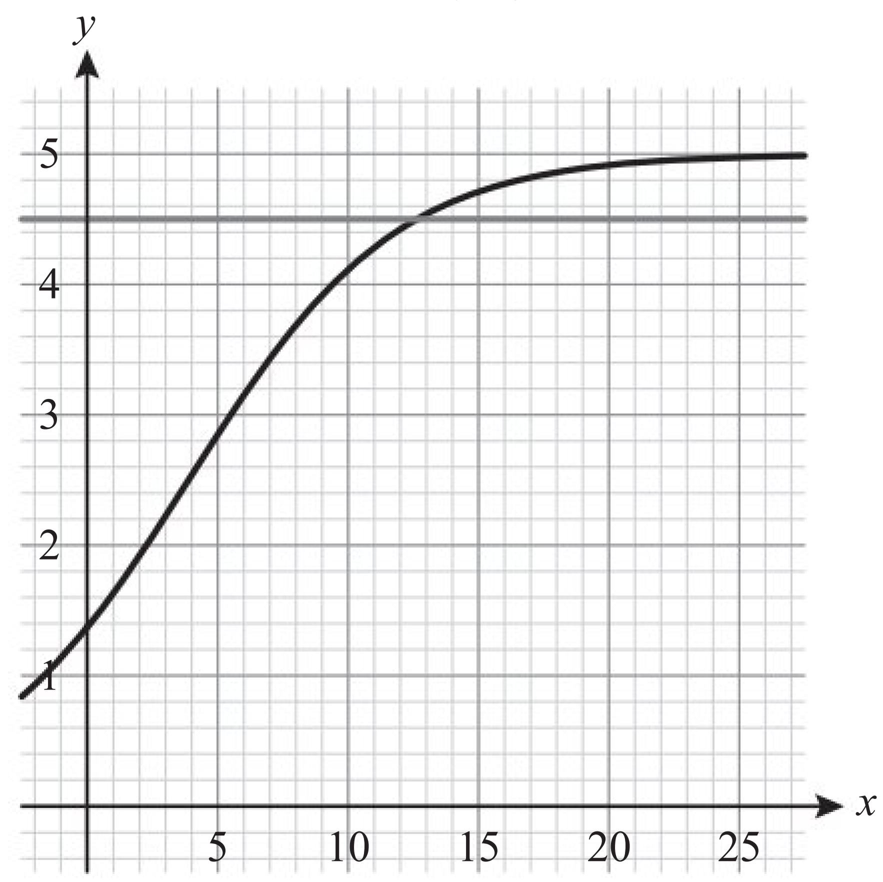
Note: do not award R0A1

[3 marks]

e i $P = \dfrac{5}{1 + 2.65e^{0}}$ M1

$P = 1370$ A1

ii Intersection of $y = \dfrac{5}{1 + 2.65e^{-0.251x}}$ and $y = 4.5$ M1



$t = 12.6$ years A1

[4 marks]

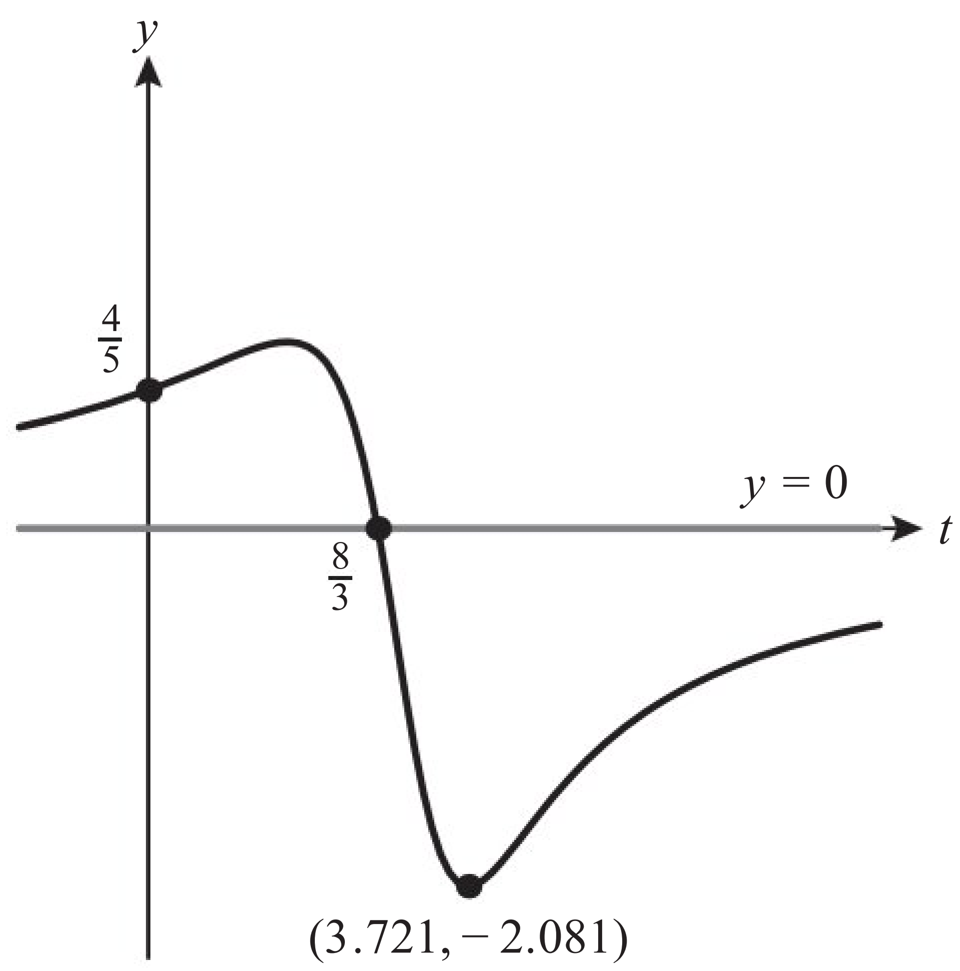
f Interpolation is required as both are outside the range of the data... R1
... so both could be unreliable/should be treated cautiously A1

[2 marks]
Total [19 marks]

5 **a** $v(0) = \frac{8}{10} = 0.8\,\text{m}\,\text{s}^{-1}$ A1

[1 mark]

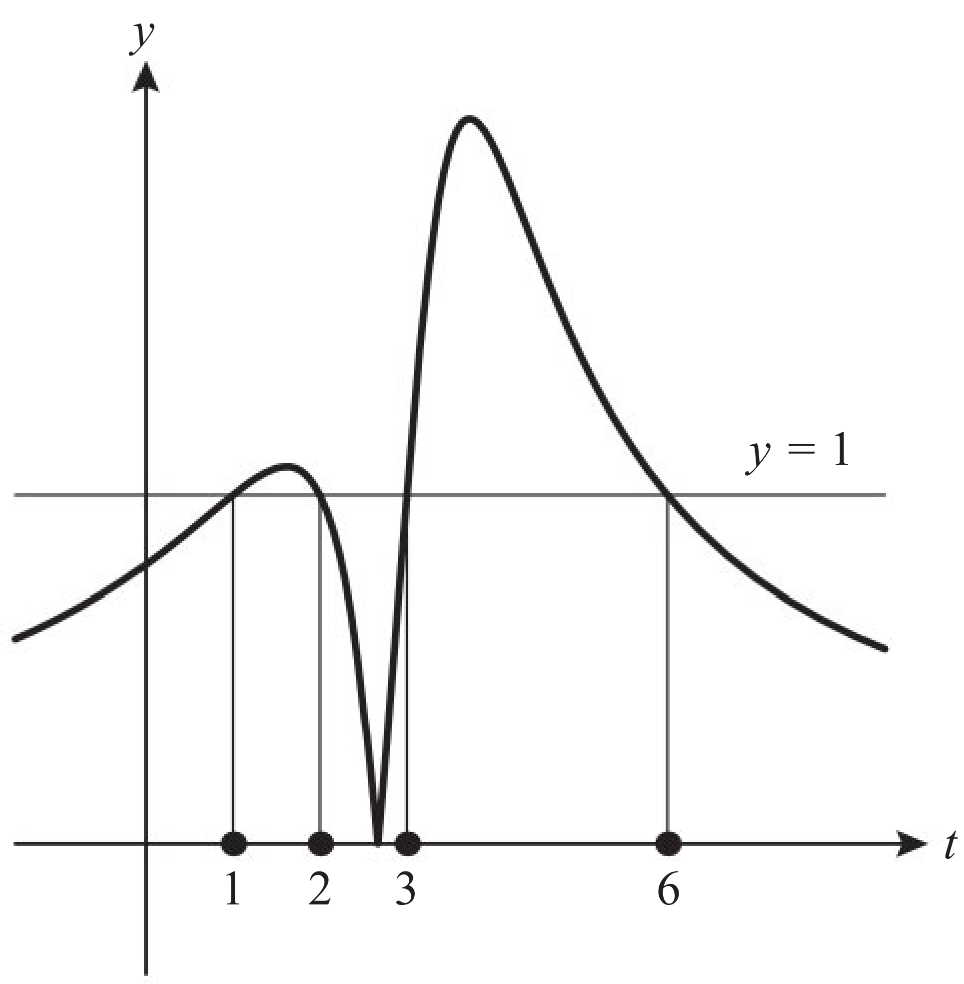
b Sketch graph $y = v(t)$ and identify minimum point (M1)



Max speed = $|-2.08| = 2.08\,\text{m}\,\text{s}^{-1}$
Note: Award M1A0 for $-2.08\,\text{m}\,\text{s}^{-1}$ A1

[2 marks]

c EITHER
$v > 1$ for $1 < t < 2$ M1
$v < 1$ for $3 < t < 6$ M1
OR
Graph $y = |v(t)|$ M1



$|v| > 1$ for $1 < t < 2$ or $3 < t < 6$ M1
So speed > 1 for 4 seconds A1

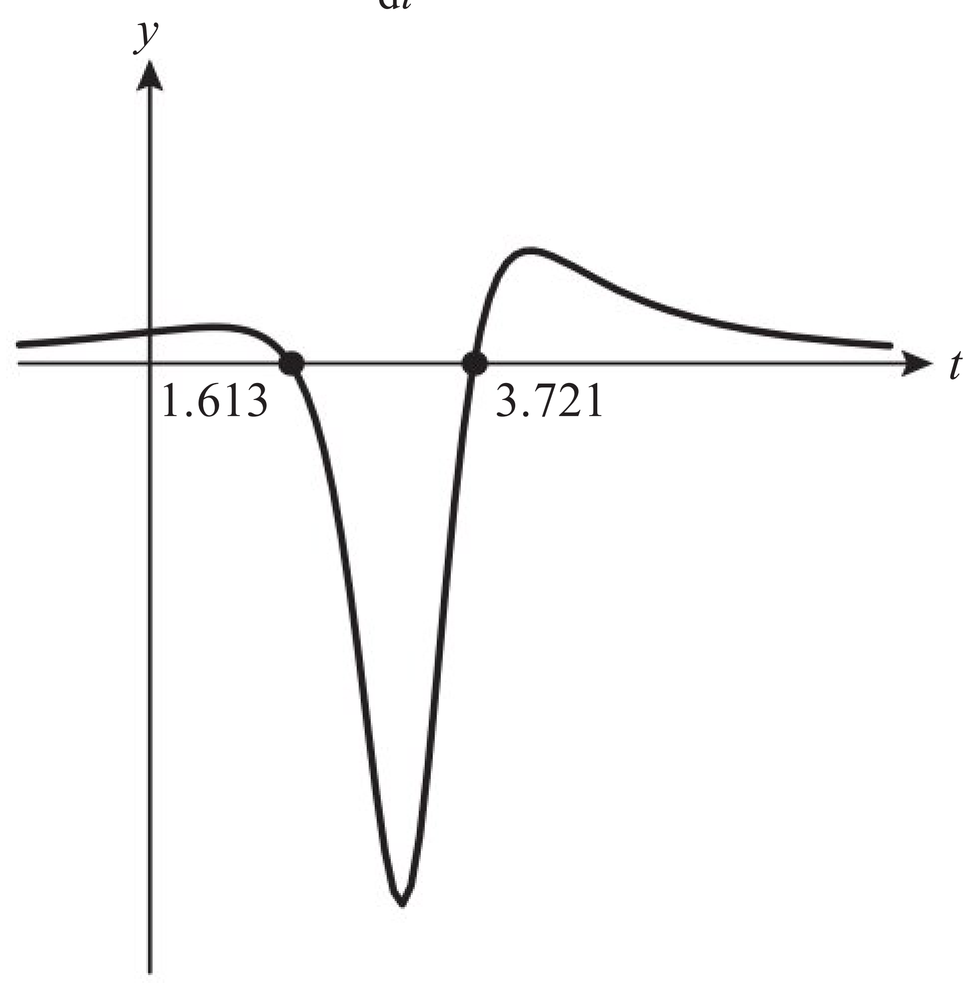
[3 marks]

d Object changes direction when $v = 0$ (M1)
$t = \frac{8}{3} = 2.67$ s A1

[2 marks]

e EITHER

Sketch graph of $y = \frac{dv}{dt}$: $y < 0$ for $1.61 < t < 3.72$ (M1)



OR

Use graph of $y = v(t)$: gradient negative for $1.61 < t < 3.72$

(between turning points) (M1)

So $a < 0$ for 2.11 seconds A1

[2 marks]

f From GDC, $\frac{dv}{dt}$ at $t = 5$... (M1)

...gives $a = 0.52\,\text{m}\,\text{s}^{-2}$ A1

[2 marks]

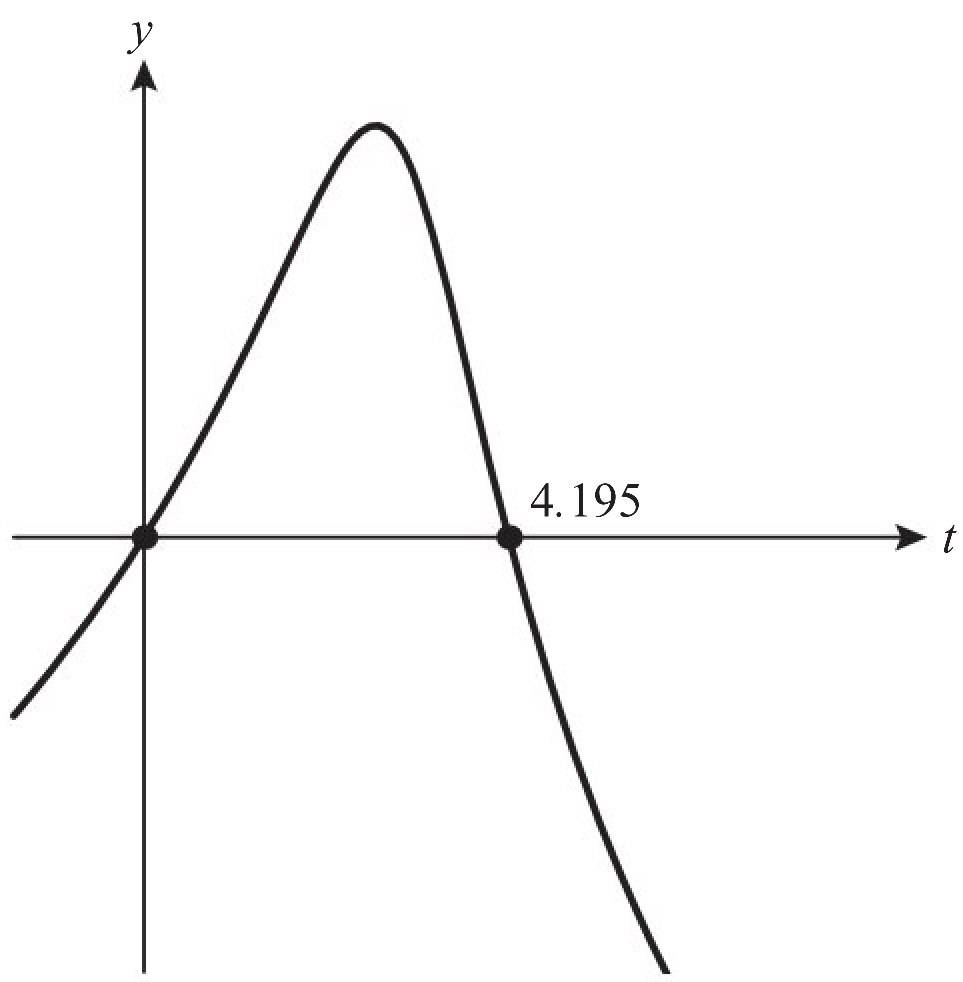
g From GDC:

distance $= \int_0^{10} \left| \frac{8 - 3t}{t^2 - 6t + 10} \right| dt$ M1

$= 9.83$ m A1

[2 marks]

h Sketch graph of $y = \int_0^x v\, dt$ (M1)



Identify x-intercept as being point at which object back at start (M1)

$t = 4.20$ seconds A1

[3 marks]

Total [17 marks]

6 a Use $\frac{dy}{dt} = -0.4x - 0.06y$ (M1)

So $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -0.4 & -0.06 \end{pmatrix}$ A1

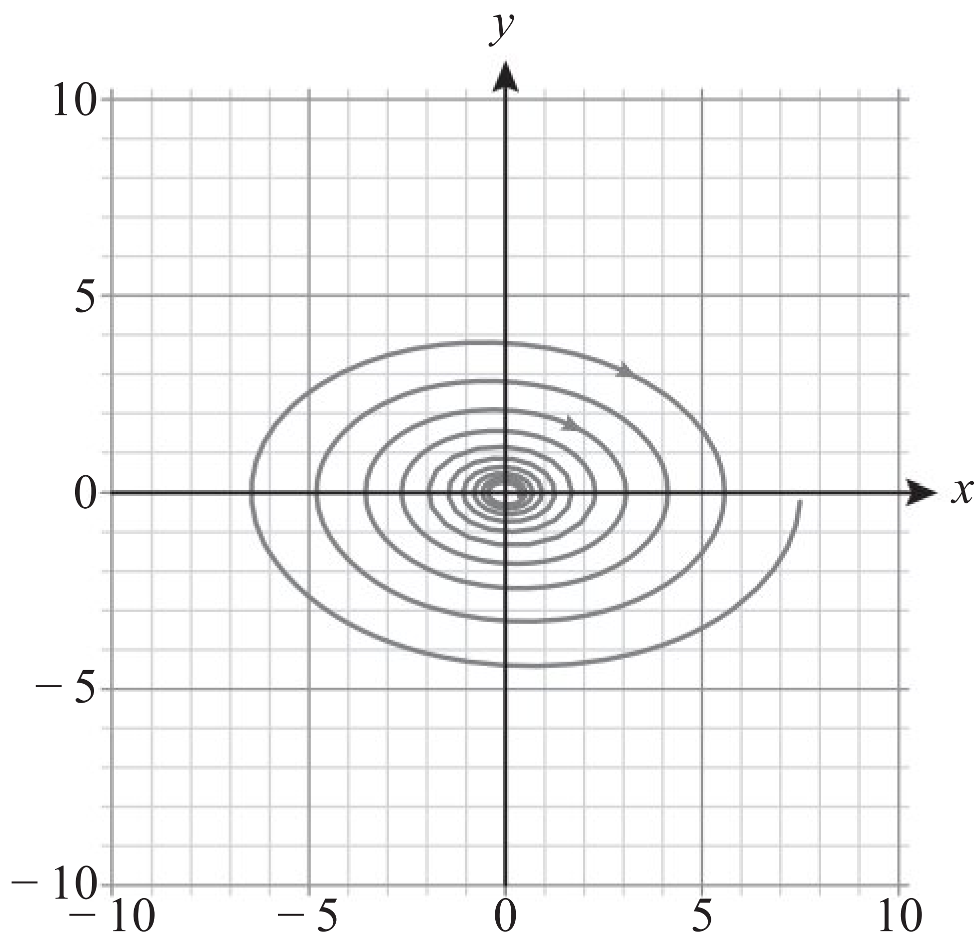
[2 marks]

b Eigenvalues: $\begin{vmatrix} -\lambda & 1 \\ -0.4 & -0.06-\lambda \end{vmatrix} = 0$ (M1)

$\lambda^2 + 0.06\lambda + 0.4 = 0$ (M1)

$\lambda = -0.03 \pm 0.632\,\mathrm{i}$ A1

Sketch: The trajectories spiral towards the origin (because the real part is negative) M1



Direction of spiral: When $x = 0$ and $y > 0$, $\frac{\mathrm{d}x}{\mathrm{d}t} = y > 0$ (x increases) M1

So spiral is clockwise A1

[6 marks]

c Use

$x_{n+1} = x_n + 0.05\,y_n$

$y_{n+1} = y_n + 0.05(-0.4x_n - 0.06y_n)$ (M1)

Initial values: $t = 2.5$, $x_0 = 0$, $y_0 = -3.8$ A1

Construct a table: M1

t	*x*	*y*
2.50	0.00	−3.80
2.55	−0.19	−3.79
2.60	−0.38	−3.77
2.65	−0.57	−3.75
2.70	−0.76	−3.73
2.75	−0.94	−3.71
2.80	−1.13	−3.68
2.85	−1.31	−3.64
2.90	−1.49	−3.60
2.95	−1.67	−3.56
3.00	−1.85	−3.52

The distance is 1.85 cm. A1

[4 marks]

d The exact solution gives $x = -1.83$ when $t = 3$ M1

So the Euler method is quite accurate. A1

[2 marks]

e Use GDC to find stationary point or solve $\frac{\mathrm{d}x}{\mathrm{d}t} = 0$ [or (4.91, −5.59) seen] M1

$t = 4.91$ A1

The distance is 5.59 cm A1

[3 marks]

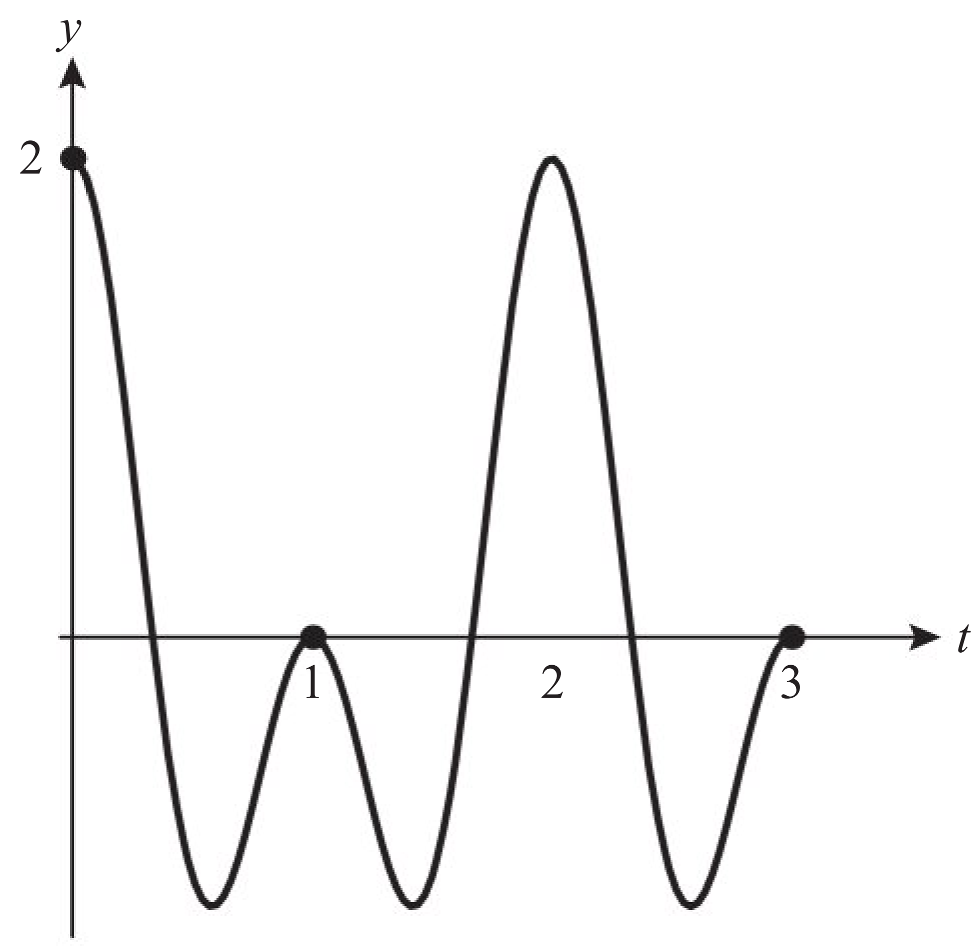
Total [17 marks]

Practice Set A: Paper 3 Mark scheme

1 **a** 2 A1

[1 mark]

b **i** A1



ii 2 A1

[2 marks]

c **i** $A = 4$ A1

$B = 8$ A1

$C = 20$ A1

ii $T = 2n$ A1

[4 marks]

d $f(t + 2n) = \cos(\pi(t + 2n)) + \cos\left(\pi\left(1 + \frac{1}{n}\right)(t + 2n)\right)$ M1

$= \cos(\pi t + 2n\pi)) + \cos\left(\left(1 + \frac{1}{n}\right)\pi t + 2\pi(n + 1)\right)$ A1

$= \cos(\pi t) + \cos\left(\left(1 + \frac{1}{n}\right)\pi t\right) = f(t)$

Since $\cos(x + 2\pi k) = \cos x$ if k is an integer. R1

[3 marks]

e **i** $\text{Re}\,(e^{(A+B)i} + e^{(A-B)i}) = \text{Re}\left(e^{Ai}(e^{Bi} + e^{-Bi})\right)$

$= \text{Re}((\cos A + i\sin A)(\cos B + i\sin B + \cos B - i\sin B))$ M1

$= \text{Re}((\cos A + i\sin A)(2\cos B))$

$= 2\cos A\cos B$ A1

ii If $P = A + B$ and $Q = A - B$ then

$A = \frac{P+Q}{2},\ B = \frac{P-Q}{2}$ M1

$\cos P + \cos Q = 2\cos\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$ A1

[4 marks]

f $f(t) = 2\cos\left(\pi\left(1 + \frac{1}{2n}\right)t\right)\cos\left(\frac{\pi}{2n}t\right)$

A1

The graph of $\cos\left(\pi\left(1 + \frac{1}{2n}\right)t\right)$ provides the high frequency oscillations. Their amplitude is determined / enveloped by the lower frequency curve $\cos\left(\frac{\pi}{2n}t\right)$ R1

[2 marks]

g $\frac{d^2x}{dt^2} = -\omega^2\cos\omega t$ M1A1

The DE becomes:

$-\omega^2\cos\omega t + 4\cos\omega t = 0$ M1

This is solved when $\omega^2 = 4$ so $\omega = 2$ A1

[4 marks]

h $\frac{d^2x}{dt^2} = -4\cos 2t - k^2\,g(k)\cos kt$ M1

The DE becomes:

$-4\cos 2t - k^2\,g(k)\cos kt + 4\cos 2t + 4g(k)\cos kt = \cos kt$ M1

$(4g(k) - k^2 g(k))\cos kt = \cos kt$

This is true for all t when $g(k)(4 - k^2) = 1$

$g(k) = \frac{1}{4-k^2}$ A1

[3 marks]

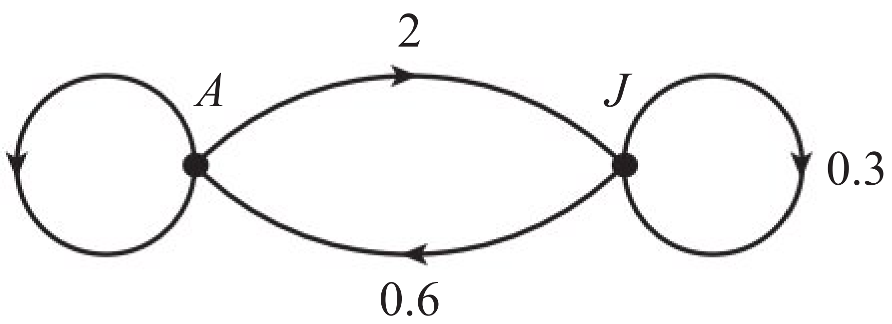
i When $k = 2$ A1

Since $\frac{1}{4-k^2} \to \infty$ as $k \to 2$ R1

[2 marks]

Total [25 marks]

2 a A2



0.5 is a measure of the survival rate of adult badgers. A1

0.6 is a measure of the rate at which juveniles mature into adults. A1

2 is the (average) number of juveniles each adult produces. A1

0.3 is a measure of the survival rate of juveniles. A1

Note: Allow some leeway in the descriptions here – for example, do not worry about people confusing rates with relative rates.

[6 marks]

b The characteristic equation is

$(0.5 - \lambda)(0.3 - \lambda) - 2 \times 0.6 = 0$ (M1)

$\lambda^2 - 0.8\lambda - 1.05 = 0$ (A1)

$\lambda_1 = \frac{3}{2},\ \lambda_2 = -\frac{7}{10}$ A1A1

[4 marks]

c When $\lambda = \frac{3}{2}$

$\begin{pmatrix} 0.5 & 0.6 \\ 2 & 0.3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 1.5\begin{pmatrix} x \\ y \end{pmatrix}$ (M1)

$0.5x + 0.6y = 1.5x$

$2x + 0.3y = 1.5y$ (M1)

Both equations are equivalent to $2x - 1.2y = 0$

$\boldsymbol{v}_1$ is therefore (anything parallel to) $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$ A1

When $\lambda = -\frac{7}{10}$

$\begin{pmatrix} 0.5 & 0.6 \\ 2 & 0.3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -0.7\begin{pmatrix} x \\ y \end{pmatrix}$ (M1)

$0.5x + 0.6y = -0.7x$

$2x + 0.3y = -0.7y$ (M1)

Both equations are equivalent to $2x + y = 0$

$\boldsymbol{v}_2$ is therefore (anything parallel to) $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ A1

[6 marks]

d **Note:** These answers will depend on the eigenvalues quoted in part **c**.

$$\begin{pmatrix} 100 \\ 20 \end{pmatrix} = \alpha\begin{pmatrix} 3 \\ 5 \end{pmatrix} + \beta\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

So $100 = 3\alpha + \beta$

$20 = 5\alpha - 2\beta$ M1

$\alpha = 20,\ \beta = 40$ A1A1

[3 marks]

e $\begin{pmatrix} A_n \\ J_n \end{pmatrix} = \boldsymbol{M}^n\begin{pmatrix} 100 \\ 20 \end{pmatrix}$ (M1)

$= \boldsymbol{M}^n(20\boldsymbol{v}_1 + 40\boldsymbol{v}_2)$ (M1)

$= 20 \times (1.5)^n\,\boldsymbol{v}_1 + 40 \times (-0.7)^n\,\boldsymbol{v}_2$ (M1)

As $n \to \infty$, $(-0.7)^n \to 0$ so (R1)

$\begin{pmatrix} A_n \\ J_n \end{pmatrix} \approx 20 \times (1.5)^n\begin{pmatrix} 3 \\ 5 \end{pmatrix}$ A1

So the long-term growth ratio is 1.5 A1

[6 marks]

f The eigenvalues are found using

$(0.5 - \lambda)(0.3 - \lambda) - x \times 0.6 = 0$ M1

$\lambda^2 - 0.8\lambda + 0.15 - 0.6x = 0$

$\lambda = \dfrac{0.8 \pm \sqrt{0.64 - 4(0.15 - 0.6x)}}{2} = \dfrac{0.8 \pm \sqrt{0.04 + 2.4x}}{2}$ A1

As seen in part **e**, the long-term growth ratio is given by the larger of the two eigenvalues. To result in decline, this must be less than 1. R1

$\dfrac{0.8 + \sqrt{0.04 + 2.4x}}{2} < 1$ M1

$0.8 + \sqrt{0.04 + 2.4x} < 2$

$\sqrt{0.04 + 2.4x} < 1.2$

$0.04 + 2.4x < 1.44$

$2.4x < 1.4$

$x < \dfrac{1.4}{2.4} = \dfrac{7}{12} \approx 0.583$ A1

[5 marks]

Total [30 marks]

Practice Set B Paper 1: Mark scheme

1 mean = 131.9, SD = 7.41 — A1
Boundaries for outliers: mean ± 2 × SD — (M1)
= 117.1, 146.7 — A1A1**ft**
147 is an outlier — A1
[5 marks]
Total [5 marks]

2 Sector area $\frac{1}{2}(7.2)^2\theta$ (= 25.92 θ) — M1
Triangle area $\frac{1}{2}(7.2)^2 \sin\theta$ (= 25.92 $\sin\theta$) — M1
$\frac{1}{2}(7.2)^2\theta - \frac{1}{2}(7.2)^2\sin\theta = 9.7$ or equivalent (e.g. $\theta - \sin\theta = 0.3742$) — A1
Solve their equation using GDC — M1
$\theta = 1.35$ — A1
[5 marks]
Total [5 marks]

3 **a** Five strips give $h = 1$ — A1
Table of values:

x	f(x)
0	0
1	0.09983
2	0.3894
3	0.7833
4	0.9996
5	0.5985

M1

0.5[0 + 0.5985 + 2(0.09983 + 0.3894 + 0.7833 + 0.9996)] — M1
= 2.57 — A1
[4 marks]

b 2.6387 — A1
[1 mark]

c $\frac{2.6387 - 2.57}{2.6387} \times 100$ — M1

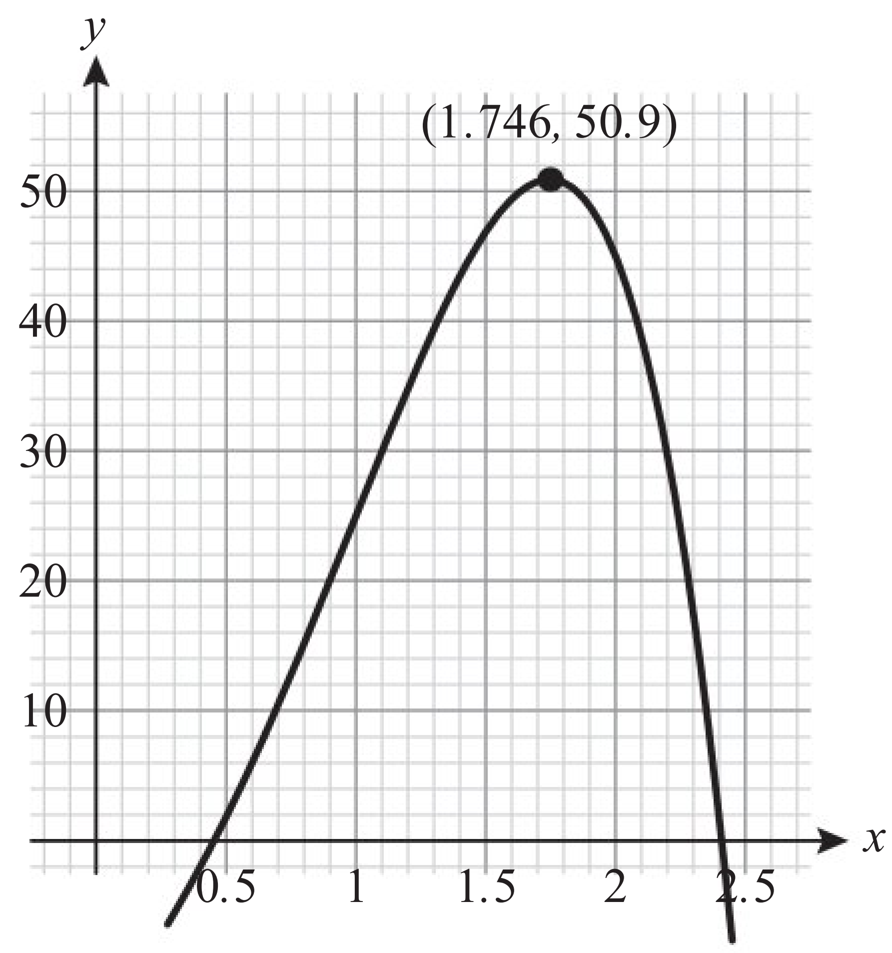
= 2.60% — A1**ft**
[2 marks]
Total [7 marks]

4 $\frac{6}{\sin(\frac{\pi}{6})} = \frac{8}{\sin(ACB)}$ — (M1)
$\sin(ACB) = \frac{2}{3}$ — A1
$ACB = 0.730$ or 2.41 (41.8° or 138°) — A1A1
$ABC = \pi - ACB - BAC$ — (M1)
= 1.89 or 0.206 (108° or 11.8°) — A1
Total [6 marks]

5 P(late) = 0.8 × 0.4 + 0.2 × 0.1 (= 0.34) — (M1)
P(late and not coffee) = 0.2 × 0.1 (= 0.02) — (M1)
P(not coffee|late) — M1
$= \frac{0.02}{0.34}$ — A1
$= \frac{1}{17}$ — A1
Total [5 marks]

6 **a** $P(x) = \int -40x^3 + 60x^2 + 30\,dx$ — M1
Note: Award M1 for evidence of integration
$= -10x^4 + 20x^3 + 30x + c$ — A1A1
Note: Award A1 for any two correct terms in x; award A1 for all terms correct and constant of integration
$45 = -10(2)^4 + 20(2)^3 + 30(2) + c$ — M1
$c = -15$
$P(x) = -10x^4 + 20x^3 + 30x - 15$ — A1
[5 marks]

b Sketch of graph (M1)



£175 A1

[2 marks]

Total [7 marks]

7 $f \circ g(x) = \dfrac{2 - \dfrac{2}{x-1}}{\dfrac{2}{x-1} + 3}$ M1

$= \dfrac{2(x-1) - 2}{2 + 3(x-1)}$ (M1)

$= \dfrac{2x-4}{3x-1}$ A1

$x = \dfrac{2y-4}{3y-1}$

$3xy - x = 2y - 4$ (M1)

$3xy - 2y = x - 4$ M1

$y = \dfrac{x-4}{3x-2}$ A1

Total [6 marks]

8 **a** Attempt to solve $2e^{-t^2} - 1 = 0$ graphically or otherwise (M1)

$t = 0.833$ s A1

[2 marks]

b $\int_0^4 2e^{-t^2} - 1 \, dt$ M1

$= -2.23$ m A1

[2 marks]

c $\int_0^4 |2e^{-t^2} - 1| \, dt$ M1

$= 3.26$ m A1

[2 marks]

Total [6 marks]

9 **a** Cars arrive independently of each other A1

Cars arrive at a constant average rate A1

[2 marks]

b $X \sim \text{Po}(14)$ (M1)

$P(X > 15) = 1 - P(X \leq 15)$ (M1)

$= 0.331$ A1

[3 marks]

c $Y \sim \text{Po}(45)$ (M1)

$P(Y \leq 39) = 0.208$ A1

[2 marks]

Total [7 marks]

10 a

	A	B	C	D	E
A	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$
B	$\frac{1}{2}$	0	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$
C	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0
D	0	0	$\frac{1}{3}$	0	0
E	$\frac{1}{2}$	0	$\frac{1}{3}$	0	0

M1A1

[2 marks]

b Raise the matrix to a large power (M1)

Each column is

$$\begin{pmatrix} 0.25 \\ 0.31 \\ 0.19 \\ 0.06 \\ 0.19 \end{pmatrix}$$

A1

The rank order is: B A1**ft**

A, C & E, D A1

[4 marks]

Total [6 marks]

11 A = –1 A1

$x = 0$: $A + B = 8$ M1

$\Rightarrow B = 9$ A1

$-1 + 9e^{-2k} = 0 \Rightarrow e^{-2k} = \frac{1}{9}$ M1

Attempt taking logarithm of both sides, e.g. $2k = -\ln\left(\frac{1}{9}\right)$ M1

$k = \ln 3$ A1

Total [6 marks]

12 a $\bar{X} \sim \mathrm{N}\left(12.6, \frac{2.8^2}{40}\right)$ M1

$\mathrm{P}(\bar{X} \le k) = 0.02$ (M1)

The critical region is $\bar{X} \le 11.7$ A1

[3 marks]

b $\bar{X} \sim \mathrm{N}\left(11.3, \frac{2.8^2}{40}\right)$ M1

$\mathrm{P}(\bar{X} > 11.7)$ M1

$= 0.183$ A1

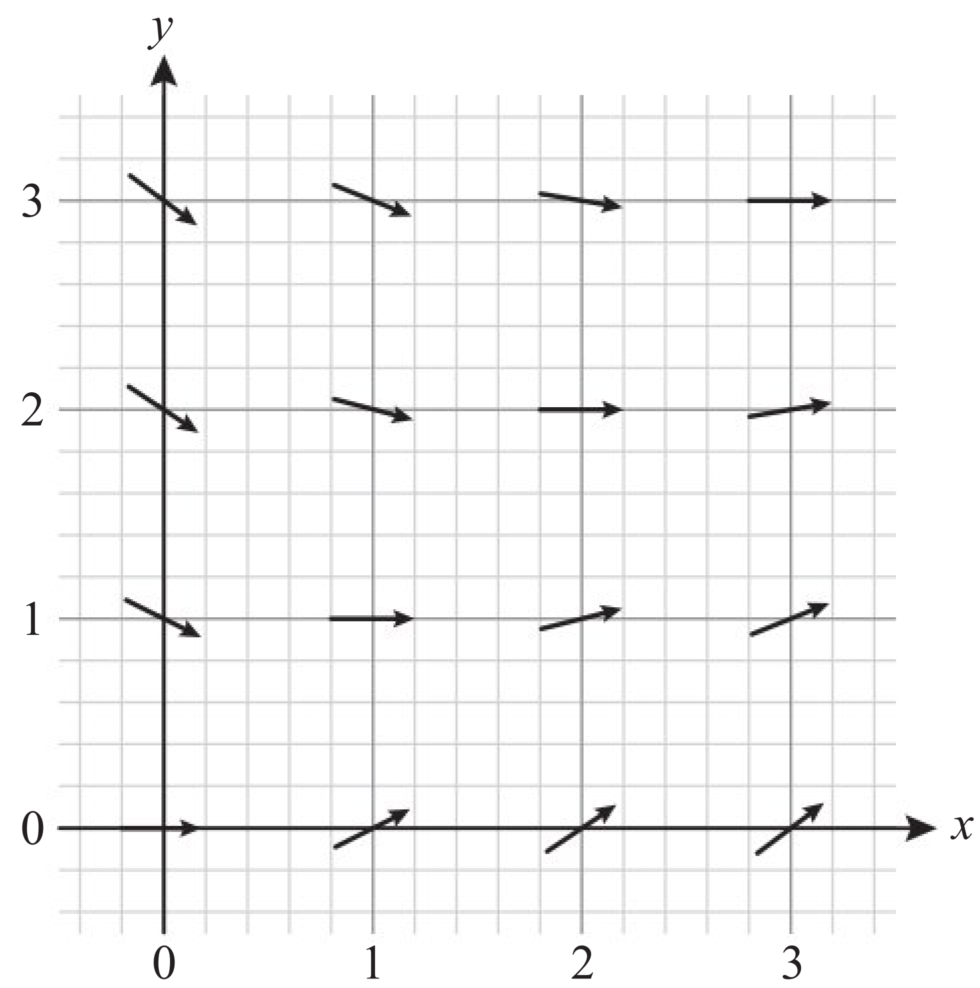
[3 marks]

Total [6 marks]

13 a

Table of values: M1

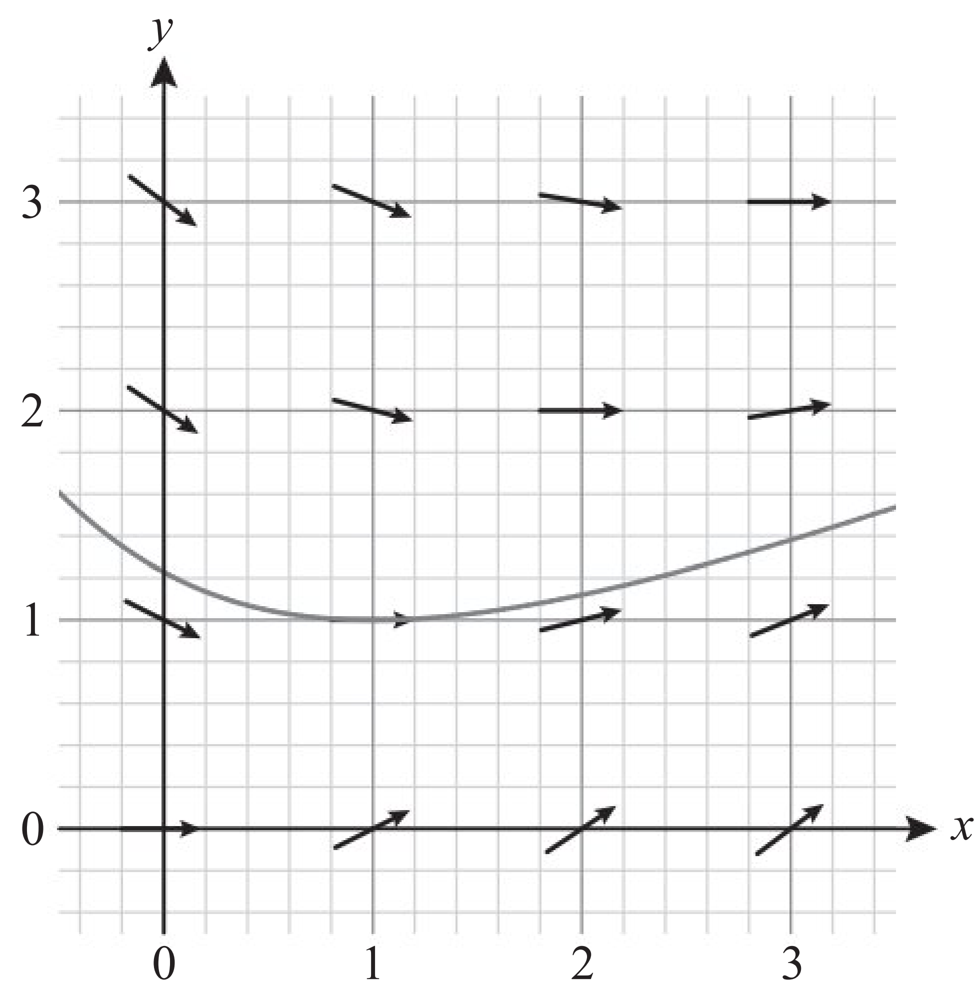
(x, y)	**0**	**1**	**2**	**3**
0	0.00	–0.50	–0.67	–0.75
1	0.50	0.00	–0.25	–0.40
2	0.67	0.25	0.00	–0.17
3	0.75	0.40	0.17	0.00



A2

[3 marks]

b



A1

[1 mark]

c Use Euler's method:

$x_{n+1} = x_n + 0.1,\ y_{n+1} = y_n + 0.1\left(\dfrac{x_n - y_n}{x_n + y_n + 1}\right)$ (M1)

x	**y**
1	1
1.1	1.000
1.2	1.003
1.3	1.009
1.4	1.018
1.5	1.029

A1

$y(1.5) \approx 1.03$ A1

[3 marks]

Total [7 marks]

14	**a**	Saddle point at (2, 3) with at least one trajectory in each quadrant	M1
		Correct direction of arrows	A1
			[2 marks]
	b	Eigenvector $\begin{pmatrix}-1\\3\end{pmatrix}$ considered or gradient –3	M1
		$\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}2\\3\end{pmatrix}+t\begin{pmatrix}-1\\3\end{pmatrix}$ or $(y-3)=-3(x-2)$	M1
		$3x+y=9$	A1
			[3 marks]
	c	e.g. It predicts that the number of flies becomes negative.	A1
			[1 mark]
	d	The number of flies increases.	A1
			[1 mark]
	e	They vary periodically / oscillate	A1
		and approach (the stable population of) 400 spiders and 100 flies.	A1
			[2 marks]
			Total [9 marks]
15	**a**	$L>2S \Leftrightarrow L-2S\ge 0$	(M1)
		$L-2S \sim N(-0.15, 0.1^2)$	A1
		$P(L-2S\ge 0)=0.0668$	A1
			[3 marks]
	b	$L>S_1+S_2 \Leftrightarrow L-S_1-S_2\ge 0$	(M1)
		$L-S_1-S_2 \sim N(-0.15, 0.0825^2)$	A1
		$P(L-S_1-S_2\ge 0)=0.0345$	A1
			[3 marks]
			Total [6 marks]
16		$h_1+h_2=\text{Im}\,(12.3e^{3.2it}+11.6e^{i(0.8+3.2t)})$	M1
		$=\text{Im}\,[e^{3.2it}(12.3+11.6e^{0.8i})]$	M1
		$=\text{Im}\,[e^{3.2it}\times 22.0e^{0.388i}]$	
		(convert to exponential form using GDC)	M1
		$=\text{Im}\,[22.0e^{i(3.2t+0.388)}]$	A1
		$=22.0\sin(3.2t+0.388)$	
		Correct amplitude	A1
		Correct $(3.2t+0.388)$	A1
			Total [6 marks]
17	**a**	$f'(t)=-L(1+Ce^{-kt})^{-2}(-Cke^{-kt})$	M1A1
		Note: Award M1 for attempt at chain rule	
		$=\dfrac{LCke^{-kt}}{(1+Ce^{-kt})^2}$	AG
			[2 marks]
	b	$f''(t)=LCk\dfrac{-ke^{-kt}(1+Ce^{-kt})^2-e^{-kt}2(1+Ce^{-kt})(-Cke^{-kt})}{(1+Ce^{-kt})^4}$	M1A1
		Note: Award M1 for attempt at quotient/product rule	
		Sets $f''(t)=0$	M1
		$LCk\dfrac{-ke^{-kt}(1+Ce^{-kt})+2Cke^{-2kt}}{(1+Ce^{-kt})^3}=0$	
		$-e^{-kt}(1+Ce^{-kt})+2Ce^{-2kt}=0$	A1
		$kt=\ln C$	M1
		$t=\dfrac{1}{k}\ln C$	A1
			[6 marks]
	c	$\dfrac{1}{k}\ln C\ge 0$	M1
		$C\ge 1$	A1
			[2 marks]
			Total [10 marks]

Practice Set B: Paper 2 Mark scheme

1 a i Arithmetic sequence, $u_1 = 30$, $d = 10$ (M1)
$u_{12} = 30 + 11 \times 10$ M1
$= 140$ A1

ii $S_{12} = 6(60 + 11 \times 10)$ or $\frac{12(30 + 140)}{2}$ M1
$= 1020$ A1

iii $\frac{N}{2}(60 + 10(N - 1)) = 2000$
OR
Create table of values M1
$N = 17.7$
OR
$S_{17} = 1870$, $S_{18} = 2070$ A1
In the 18th month A1

[8 marks]

b i Geometric sequence, $u_1 = 30$, $r = 1.1$ (M1)
$S_{12} = \frac{30(1.1^{12} - 1)}{1.1 - 1}$ M1
$= 642$ A1

ii $30 \times 1.1^{N-1} > 1000$ M1
$N = 37.8$ (M1)
In the 38th month A1

[6 marks]

c i Multiply answer to **aii** or **bi** by the profit at least once M1
Stella: $1020 \times 2.20 = £2244$ A1
Giulio: $642 \times 3.10 = £1990$ A1

ii $\frac{30(1.1^N - 1)}{0.1} \times 3.10 > \frac{N}{2}(60 + 10(N - 1)) \times 2.20$ M1
$N = 21.4$ (M1)
In the 22nd month A1

[6 marks]

Total [20 marks]

2 a There are fewer assumptions in using a paired test A1
A paired test eliminates variation between individuals A1

[2 marks]

b i $H_0: \mu_B = \mu_A$
$H_1: \mu_B > \mu_A$ A1

ii Finds differences between values before and after (M1)
$p = 0.0197$ A1

iii $0.0197 < 0.05$ R1
Reject the null hypothesis; there is sufficient evidence at the 5% level of a decrease in the mean level of cholesterol A1

Note: Award R1 for a correct comparison of their correct p-value to the test level, award A1 for the correct result from that comparison. Do not award R0A1.

[5 marks]

c i 5.35 A1

ii 36.1 A1

[2 marks]

d i H_0: The data follow a normal distribution
H_1: The data do not follow a normal distribution A1

ii A2

Difference	$d < -5$	$-5 \le d < 0$	$0 \le d < 5$	$5 \le d < 10$	$10 \le d < 15$	$d \ge 15$
Expected frequency	3.95	13.41	26.98	28.24	15.38	5.04

Note: Award A2 for all 6 correct expected values, A1 for 4 or 5 correct values, A0 otherwise.

iii Combining first two columns (M1)
Degrees of freedom $= 5 - 2 - 1 = 2$ A1

iv $p = 0.562$ A2

v $0.562 > 0.1$ R1

Do not reject the null hypothesis; there is insufficient evidence at the 10% level that the data do not follow a normal distribution A1

Note: Award R1 for a correct comparison of their correct p-value to the test level, award A1 for the correct result from that comparison. Do not award R0A1.

[9 marks]

e Yes, since the differences need to be normally distributed for the paired t-test to be valid R1

[1 mark]

Total [19 marks]

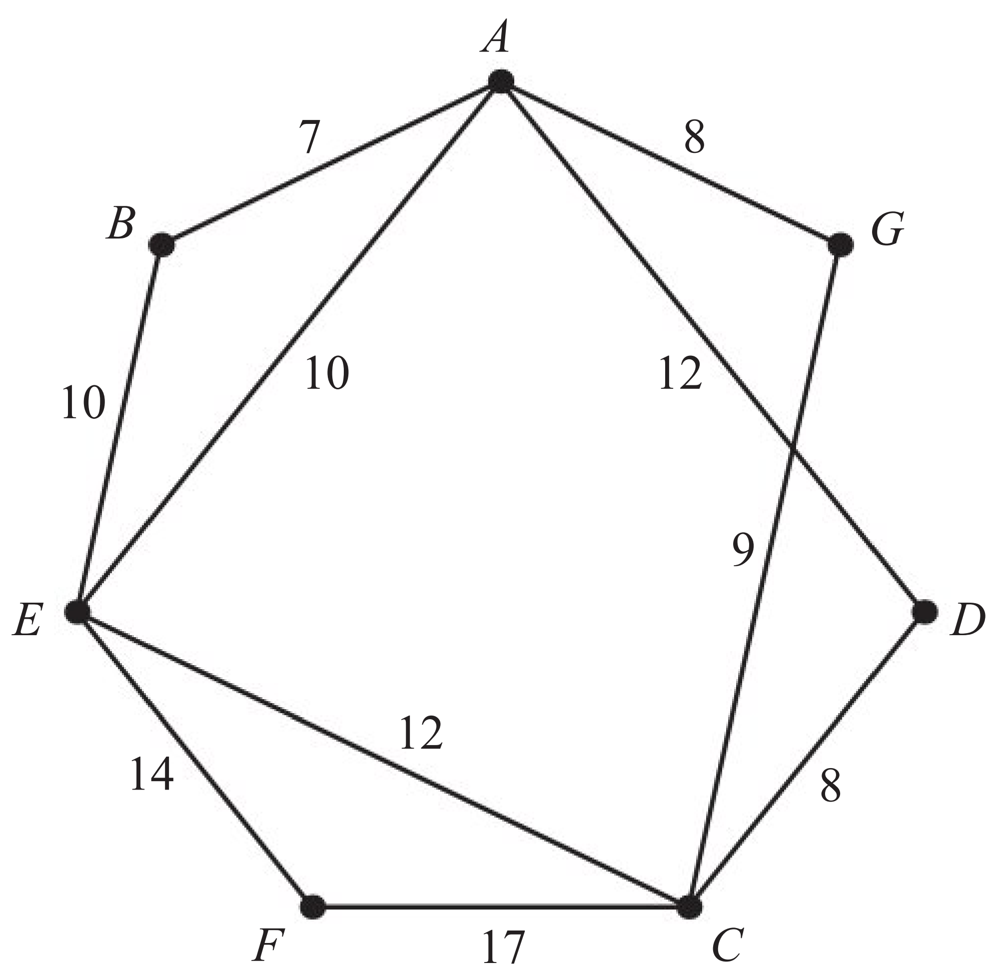
3 a Award 1 mark for at least two correct entries

Vertex	A	B	C	D	E	F	G
Degree	4	2	4	2	4	2	2

A1A1

[2 marks]

b



Award 1 mark for correct connections and 1 mark for correct numbers. A1A1

[2 marks]

c Every vertex has even degree R1

Attempt to add the weights of all the edges (M1)

107 km A1

[3 marks]

d Include GC (9) M1

CD(8) + DA(12) = 20

CE(12) + EA(10) = 22 M1

Repeat edges GC, CD and DA A1

Length = [(107 − 8) + (9 + 20)] = 128 km A1

[4 marks]

e Add edges, starting with AB(7), AG(8), CD(8) and CG(9) M1

Add AE or BE (10) M1

Skip CE; add EF(14) A1

The length of the cable required is 56 km A1

[4 marks]

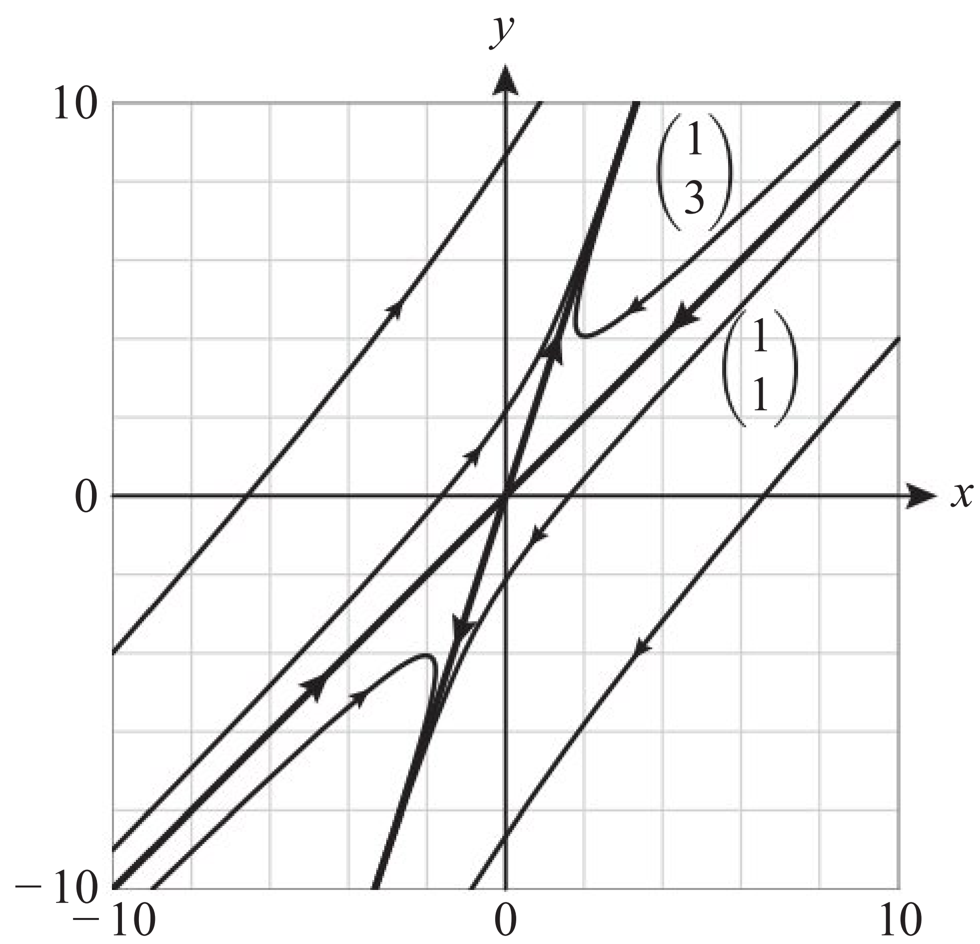
Total [15 marks]

4 a $\begin{pmatrix}-5 & 2\\-6 & 3\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}-3\\-3\end{pmatrix}, \begin{pmatrix}-5 & 2\\-6 & 3\end{pmatrix}\begin{pmatrix}1\\3\end{pmatrix}=\begin{pmatrix}1\\3\end{pmatrix}$ M1A1

The eigenvalues are −3 and 1 A1

[3 marks]

b



Eigenvector directions shown	A1
Trajectories OR stating that the origin is a saddle point	A1
Correct directions of trajectories	A1
	[3 marks]

c $\begin{pmatrix}x\\y\end{pmatrix} = Ae^{-3t}\begin{pmatrix}1\\1\end{pmatrix} + Be^{t}\begin{pmatrix}1\\3\end{pmatrix}$

e^{-3t} and e^{t} terms	M1
Eigenvectors and constants in correct place	A1
	[2 marks]

d

Using $x = 1, y = 2, t = 0$:	
$(A + B = 1, A + 3B = 2)$	(M1)
Solve the equations $\left(A = \frac{1}{2}, B = \frac{1}{2}\right)$	M1
$x = \frac{1}{2}e^{-3t} + \frac{1}{2}e^{t}, y = \frac{1}{2}e^{-3t} + \frac{3}{2}e^{t}$	A1A1
Both increase without a limit.	A1
	[5 marks]

e i

Using $x = 2, y = 1, t = 0$:	
$(A + B = 2, A + 3B = 1)$	(M1)
Solve the equations $\left(A = \frac{5}{2}, B = -\frac{1}{2}\right)$	M1
$\left[x = \frac{5}{2}e^{-3t} - \frac{1}{2}e^{t}, y = \frac{5}{2}e^{-3t} - \frac{3}{2}e^{t}\right]$	
Set $y = 0$: $y = \frac{5}{2}e^{-3t} - \frac{3}{2}e^{t}$	M1
$t = 0.128$	A1
(When $t = 0.128, x = 1.14$) So 114 predators at that time.	A1

ii

Attempt to solve $\frac{dx}{dt} = -5x$ (e.g. separate variables or state exponential decay)	(M1)
Use $t = 0.128, x = 1.14$	M1
$x = 2.16e^{-5t}$	A1
	[8 marks]
	Total [21 marks]

5 a

$\mathbf{v} = c_1\mathbf{i} + (c_2 - 9.8t)\mathbf{j}$	M1
When $t = 0$, $\mathbf{v} = 8\mathbf{i} + 14\mathbf{j}$: $8\mathbf{i} + 14\mathbf{j} = c_1\mathbf{i} + c_2\mathbf{j}$	M1
$\mathbf{v} = 8\mathbf{i} + (14 - 9.8t)\mathbf{j}$	A1
	[3 marks]

b

$\mathbf{r} = (8t + c_1)\mathbf{i} + (14t - 4.9t^2 + c_2)\mathbf{j}$	M1
When $t = 0$, $\mathbf{r} = 0\mathbf{i} + 0\mathbf{j}$: $0\mathbf{i} + 0\mathbf{j} = c_1\mathbf{i} + c_2\mathbf{j}$	M1
$\mathbf{r} = 8t\mathbf{i} + (14t - 4.9t^2)\mathbf{j}$	A1
$OQ = 2PQ$: $8t = 2(14t - 4.9t^2)$	M1
$t = 2.04$ s	A1
	[5 marks]

	c	When $t = 2.04$, $\mathbf{v} = 8\mathbf{i} - 6\mathbf{j}$	M1
		speed $= \sqrt{8^2 + (-6)^2} = 10\,\text{m}\,\text{s}^{-1}$	A1
			[2 marks]
	d	$14 - 9.8t = 6$	M1
		$t = 0.816\,\text{s}$	A1
			[2 marks]
	e	$\mathbf{r} = 12(t-1)\mathbf{i} + (k(t-1) - 4.9(t-1)^2)\mathbf{j}$	M1A1
		For collision:	
		$\begin{cases} 8t = 12(t-1) \ (1) \\ 14t - 4.9t^2 = k(t-1) - 4.9(t-1)^2 \ (2) \end{cases}$	M1
		From (1): $t = 3$	A1
		Substitute their value of t into (2):	M1
		$k = 8.75$	A1
			[6 marks]
			Total [18 marks]
6	**a**	$\begin{pmatrix} 0.85 & 0.25 \\ 0.15 & 0.75 \end{pmatrix}$	M1A1
			[2 marks]
	b	$\begin{pmatrix} 0.85 & 0.25 \\ 0.15 & 0.75 \end{pmatrix}^4 \begin{pmatrix} 9000 \\ 7000 \end{pmatrix} = \begin{pmatrix} 9870 \\ 6130 \end{pmatrix}$	M1A1
			[2 marks]
	c	$\begin{vmatrix} 0.85 - \lambda & 0.25 \\ 0.15 & 0.75 - \lambda \end{vmatrix} = 0$	M1
		$\lambda = 1, 0.6$	A1
		$\begin{pmatrix} 5 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}$	(M1)A1
			[4 marks]
	d	EITHER	
		$\mathbf{P} = \begin{pmatrix} 5 & 1 \\ 3 & -1 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} 1 & 0 \\ 0 & 0.6 \end{pmatrix}$	A1A1
		OR	
		$\mathbf{P} = \begin{pmatrix} 1 & 5 \\ -1 & 3 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} 0.6 & 0 \\ 0 & 1 \end{pmatrix}$	A1A1
			[2 marks]
	e	$\mathbf{P}^{-1} = \frac{1}{8}\begin{pmatrix} 1 & 1 \\ 3 & -5 \end{pmatrix}$	A1
		$\begin{pmatrix} 5 & 1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0.6^n \end{pmatrix}\frac{1}{8}\begin{pmatrix} 1 & 1 \\ 3 & -5 \end{pmatrix}\begin{pmatrix} 9000 \\ 7000 \end{pmatrix}$	M1A1
		Number of subscribers for $S = 10\,000 - 1000 \times 0.6^n$	M1A1
			[5 marks]
	f	10 000	A1
			[1 mark]
	g	Does not take into account people who might not have subscription television to start with or who want to revert to not having it.	A1
			[1 mark]
			Total [17 marks]

Practice Set B Paper 3: Mark scheme

1 a $\bar{X} = \frac{X_1 + X_2}{2}$ A1

[1 mark]

b $E(\bar{X}) = E\left(\frac{1}{2}X_1 + \frac{1}{2}X_2\right) = \frac{1}{2}E(X_1) + \frac{1}{2}E(X_2)$ M1

$= \frac{1}{2}\mu + \frac{1}{2}\mu$ A1

$= \mu$ AG

$\text{Var}(\bar{X}) = \text{Var}\left(\frac{1}{2}X_1 + \frac{1}{2}X_2\right) = \frac{1}{4}\text{Var}(X_1) + \frac{1}{4}\text{Var}(X_2)$ M1

$= \frac{1}{4}\sigma^2 + \frac{1}{4}\sigma^2$

$= \frac{1}{2}\sigma^2$ A1

[4 marks]

c i $E(X^2) = \text{Var}(X) + E(X)^2$ A1

ii $E(S^2) = E\left(\frac{X_1^2 + X_2^2}{2} - \bar{X}^2\right) = \frac{1}{2}E(X_1^2) + \frac{1}{2}E(X_2^2) - E(\bar{X}^2)$ M1

$= \frac{1}{2}(\text{Var}(X_1) + E(X_1)^2) + \frac{1}{2}(\text{Var}(X_1) + E(X_1)^2) - (\text{Var}(\bar{X}) + E(\bar{X})^2)$ M1

$= \frac{1}{2}(\sigma^2 + \mu^2) + \frac{1}{2}(\sigma^2 + \mu^2) - \left(\frac{1}{2}\sigma^2 + \mu^2\right)$ A1

$= \frac{1}{2}\sigma^2$ AG

[4 marks]

d i $E(M) = \frac{2}{5}E(X_1) + \frac{3}{5}E(X_2)$ M1

$= \frac{2}{5}\mu + \frac{3}{5}\mu$ A1

$= \mu$ AG

ii $\text{Var}(M) = \frac{4}{25}\text{Var}(X_1) + \frac{9}{25}\text{Var}(X_2)$ M1

$= \frac{13}{25}\sigma^2$ A1

$> \frac{1}{2}\sigma^2$ therefore $\bar{X}$ is a more efficient estimator. R1

[5 marks]

e i $L = P(Y = a)P(Y = b)$ M1

$= p(1-p)^{a-1} \times p(1-p)^{b-1}$ A1

ii $L = p^2(1-p)^{a+b-2}$

$\frac{dL}{dp} = 2p(1-p)^{a+b-2} - (a+b-2)p^2(1-p)^{a+b-3}$ M1A1

At a max, $\frac{dL}{dp} = 0$ M1

$p(1-p)^{a+b-3}(2(1-p) - (a+b-2)p) = 0$ M1

Since $p \neq 0$ and $p \neq 1$ at the maximum value of L R1

$2 - 2p = ap + bp - 2p$

$2 = ap + bp$

$p = \frac{2}{a+b}$ A1

[8 marks]

f i $S^2 = \frac{4^2 + 8^2}{2} - 6^2 = 4$ M1

Unbiased estimate of $\sigma^2 = 2S^2 = 8$ A1

ii $p = \frac{2}{4+8} = \frac{1}{6}$ A1

[3 marks]

Total [25 marks]

2 a $|e^{\frac{2\pi i}{3}} - 1| = |\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} - 1|$ M1

$= \left|-\frac{1}{2} + \frac{\sqrt{3}}{2}i - 1\right|$ A1

$= \sqrt{\left(-\frac{3}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$

$= \sqrt{3}$ A1

[3 marks]

b Bill: $e^{\frac{2\pi i}{3}}$ A1
Charlotte: $e^{\frac{4\pi i}{3}}$ A1
[2 marks]

c Using part **a**: $\sqrt{3}$ units in $\sqrt{3}$ seconds. A1A1
[2 marks]

d The direction from Z_A to Z_B is $Z_B - Z_A$ R1
The distance travelled per unit time is one, so this is $\frac{Z_B - Z_A}{|Z_B - Z_A|}$ R1
[2 marks]

e $Z_B = e^{\frac{2\pi i}{3}} Z_A$ A1
[1 mark]

f $\frac{dZ_A}{dt} = \frac{dr}{dt}e^{i\theta} + ire^{i\theta}\frac{d\theta}{dt}$ M1A1
[2 marks]

g $\frac{dr}{dt}e^{i\theta} + ire^{i\theta}\frac{d\theta}{dt} = \frac{e^{\frac{2\pi i}{3}}Z_A - Z_A}{\left|e^{\frac{2\pi i}{3}}Z_A - Z_A\right|} = \frac{Z_A\left(e^{\frac{2\pi i}{3}} - 1\right)}{|Z_A|\left|e^{\frac{2\pi i}{3}} - 1\right|}$ M1A1

$= \frac{re^{i\theta}\left(e^{\frac{2\pi i}{3}} - 1\right)}{r\left|e^{\frac{2\pi i}{3}} - 1\right|}$ M1A1

$= \frac{e^{i\theta}}{\sqrt{3}}\left(-\frac{3}{2} + \frac{i\sqrt{3}}{2}\right)$ A1

$= e^{i\theta}\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$

Dividing through by $e^{i\theta}$:

$\frac{dr}{dt} + ir\frac{d\theta}{dt} = -\frac{\sqrt{3}}{2} + \frac{1}{2}i$

Comparing real and imaginary parts:

$\frac{dr}{dt} = -\frac{\sqrt{3}}{2}$ A1

$r\frac{d\theta}{dt} = \frac{1}{2}$ A1
[7 marks]

h $r = -\frac{\sqrt{3}}{2}t + c$ M1

When $t = 0$, $r = 1$ so $c = 1$ M1

$r = 1 - \frac{\sqrt{3}}{2}t$ A1

$\frac{d\theta}{dt} = \frac{1}{2\left(1 - \frac{\sqrt{3}}{2}t\right)} = \frac{1}{2 - \sqrt{3}t}$ M1

$\theta = \frac{1}{\sqrt{3}}\ln(2 - \sqrt{3}t) + c$ A1

When $t = 0$, $\theta = 0$ so $c = \frac{1}{\sqrt{3}}\ln 2$ M1

$\theta = -\frac{1}{\sqrt{3}}\ln\left(\frac{2}{2 - \sqrt{3}t}\right)$ A1
[7 marks]

i Meet when $r = 0$ M1
This happens when $1 - \frac{\sqrt{3}}{2}t = 0$

So $t = \frac{2}{\sqrt{3}}$ A1

Since $v = 1$ the distance travelled is $\frac{2}{\sqrt{3}}$ units. A1

As $t \to \frac{2}{\sqrt{3}}$, $\theta \to \infty$ so the snails make an infinite number of rotations A1
[4 marks]
Total [30 marks]

Practice Set C Paper 1: Mark scheme

1 $a + 4d = 7$, $a + 9d = 81$ M1A1

Solving:

$a = -52.2$, $d = 14.8$ A1

$S_{20} = \frac{20}{2}(-104.4 + 19 \times 14.8)$ (M1)

$= 1768$ A1

Total [5 marks]

2 **a** Stratified sampling A1

[1 mark]

b Correct regression line attempted M1

$y = -1.33x + 6.39$ A1

[2 marks]

c For every extra hour spent on social media, 1.33 hours less spent on homework. A1

No social media gives around 6.39 hours for homework. A1

[2 marks]

Total [5 marks]

3 **a** $\frac{4}{3}\pi(3^3) \times 1.45$ (M1)

$= 164$ g A1

[2 marks]

b Each volume [mass] is $\frac{1}{8}$ the previous one A1

Sum to infinity $= \frac{164}{1 - \frac{1}{8}} = 187$ g M1A1

Hence the mass is always smaller than 200 g A1

[4 marks]

Total [6 marks]

4 **a** $\frac{1}{4} + k + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = 1$ M1

$k = \frac{5}{16}$ A1

[2 marks]

b $E(G) = \left(0 \times \frac{1}{4}\right) + \left(1 \times \frac{5}{16}\right) + \left(2 \times \frac{1}{4}\right) + \left(3 \times \frac{1}{8}\right) + \left(4 \times \frac{1}{16}\right) = \frac{23}{16}$ M1A1

Their $E(G)$ multiplied by 38 M1

$= 54.6$ A1

[4 marks]

Total [6 marks]

5 Write $z = x + iy$ (M1)

Then $3x + 3iy - 4x + 4iy = 18 + 21i$ A1

Compare real and imaginary parts M1

$z = -18 + 3i$ A1

$\left|\frac{z}{3}\right| = \sqrt{6^2 + 1^2}$ M1

$= \sqrt{37}$ A1

Total [6 marks]

6 **a** Translation 2 left A1

[1 mark]

b $y = 0$ A1

$x = -2$ A1

[2 marks]

c $x = \frac{1}{y + 2}$ (M1)

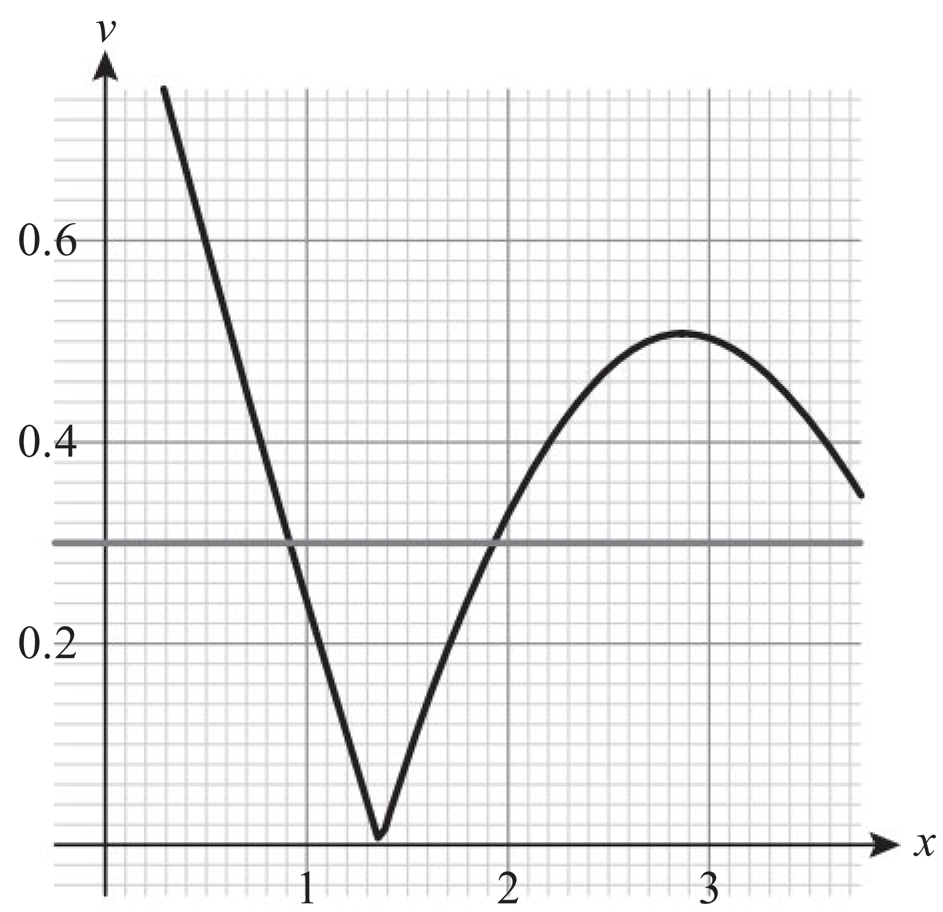
$x(y + 2) = 1$ (M1)

Note: The first method mark for switching x and y can be awarded before or after the second method mark.

$y = \frac{1}{x} - 2 = f^{-1}(x)$ A1

[3 marks]

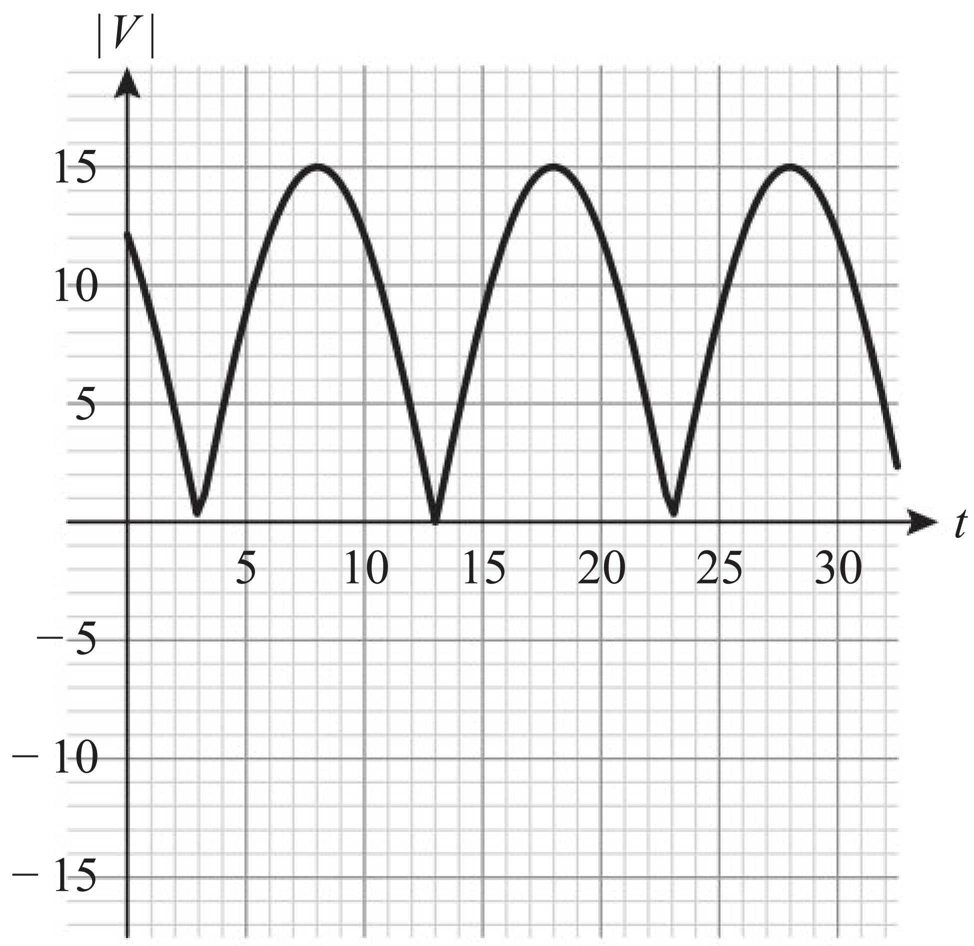
	d	$y = -2$	A1
		$x = 0$	A1
			[2 marks]
			Total [8 marks]
7	**a**	A	A1
		Gradient is zero and changing from positive to negative	A1
			[2 marks]
	b	B, D	A1
		and E	A1
		Second derivative is zero and changes sign	A1
			[3 marks]
			Total [5 marks]
8		$\|\mathbf{a}\|\,\|\mathbf{b}\| \cos\theta = 17$	M1
		$\|\mathbf{a}\|\,\|\mathbf{b}\| \sin\theta = \sqrt{4+1+25}$ [$= \sqrt{30}$]	M1
		$\tan\theta = \frac{\sqrt{30}}{17}$	M1A1
		$\theta = 17.9°$	A1
			Total [5 marks]
9	**a**	Integrate $\|v\|$	(M1)
		With limits 0 and 5	(M1)
		Distance = 1.8 m	A1
			[3 marks]
	b	Sketch $\left\|\frac{dv}{dt}\right\|$ [or $\frac{dv}{dt}$]	(M1)



		Intersect with $y = 0.3$ [or with both 0.3 and –0.3]	(M1)
		$t = 0.902$ and 1.93 seconds	A1
			[3 marks]
			Total [6 marks]
10	**a**	The underlying population distributions of lifetimes need to be normal.	A1
			[1 mark]
	b	H_0: The two population means are equal	
		H_1: The two population means are different	A1
		Using two-sample t-test (pooled)	M1
		$p = 0.0263$	A1
		<0.05	M1
		Sufficient evidence that the population mean lifetimes are different.	A1
			[5 marks]
			Total [6 marks]
11	**a**	$M = \frac{1}{2}((8+5)\mathbf{i} + (-3+1)\mathbf{j})$	M1
		$= 6.5\mathbf{i} - \mathbf{j}$	A1
			[2 marks]
	b	$\overrightarrow{AB} = -3\mathbf{i} + 4\mathbf{j}$	A1
		Attempt to find vector perpendicular to $\overrightarrow{AB}$: $4\mathbf{i} - 3\mathbf{j}$	M1
		$\mathbf{r} = 6.5\mathbf{i} - \mathbf{j} + \lambda(4\mathbf{i} - 3\mathbf{j})$	A1
			[3 marks]

	c	$\lvert\mathbf{4i} - 3\mathbf{j}\rvert = \sqrt{4^2 - (-3)^2} = 5$	(M1)
		$\mathbf{v} = \frac{20}{5}(4\mathbf{i} - 3\mathbf{j}) = 16\mathbf{i} - 12\mathbf{j}$	A1
		$6.5\mathbf{i} - \mathbf{j} = a\mathbf{i} + b\mathbf{j} + 0.2(16\mathbf{i} - 12\mathbf{j})$	(M1)
		$\mathbf{r} = 3.3\mathbf{i} + 1.4\mathbf{j} + t(16\mathbf{i} - 12\mathbf{j})$	A1
			[4 marks]
			Total [9 marks]
12	**a**	Use $\bar{X} \approx$ normal	M1
		mean = 7.6	A1
		variance = $\frac{3.7}{40}$	A1
		$P(\bar{X} > 8) = 0.0942$	A1
			[4 marks]
	b	We are not told that the population distribution is normal.	A1
			[1 mark]
			Total [5 marks]
13	**a**	Substitute: $3.6 \times 10^{-6} = \frac{k}{0.002^2}$	M1
		$k = 1.44 \times 10^{-11}$	A1
			[2 marks]
	b	Use $\frac{dr}{dt} = 0.07$	A1
		Use $\frac{dF}{dr} = -\frac{2k}{r^3}$	M1
		$\frac{dF}{dt} = -\frac{2k}{r^3}\frac{dr}{dt}$	A1
		$= 2.52 \times 10^{-4}$ (N s^{-1})	A1
			[4 marks]
			Total [6 marks]
14	**a**	$\begin{pmatrix} 0.5 & 0.1 & 0 \\ 0.45 & 0.5 & 0.1 \\ 0.05 & 0.4 & 0.9 \end{pmatrix}$	M1A1
			[2 marks]
	b	$\begin{pmatrix} 0.5 & 0.1 & 0 \\ 0.45 & 0.5 & 0.1 \\ 0.05 & 0.4 & 0.9 \end{pmatrix}\begin{pmatrix} G \\ S \\ T \end{pmatrix} = \begin{pmatrix} G \\ S \\ T \end{pmatrix}$	(M1)
		$0.5G + 0.1S = G$	
		$0.45G + 0.5S + 0.1T = S$	A1
		$0.05G + 0.4S + 0.9T = T$	
		$G + S + T = 1$	A1
			[3 marks]
	c	Reduce to system of three equations in three unknowns	M1
		$G = \frac{2}{53}, S = \frac{10}{53}, T = \frac{41}{53}$	A1
			[2 marks]
			Total [7 marks]
15		Separate variables and attempt to integrate:	
		$\int dy = \int \frac{4x}{3x^2 + 1} dx$	M1
		Use substitution $u = 3x^2 + 1$	(M1)
		Obtain $k \ln(3x^2 + 1)$	A1
		$y = \frac{2}{3}\ln(3x^2 + 1) + c$	A1
		Use $x = 0, y = 1$	M1
		Obtain $c = 1$	A1
			Total [6 marks]
16	**a**	$a = 15$	A1
		$b = \frac{2\pi}{\text{period}}$	M1
		$= 0.314$	A1
		$c = 3$	A1
			[4 marks]

b A1



[1 mark]

Total [5 marks]

17 a Two equations, with the first one correct M1

$\frac{dx}{dt} = y, \frac{dy}{dt} = -0.49x$ A1

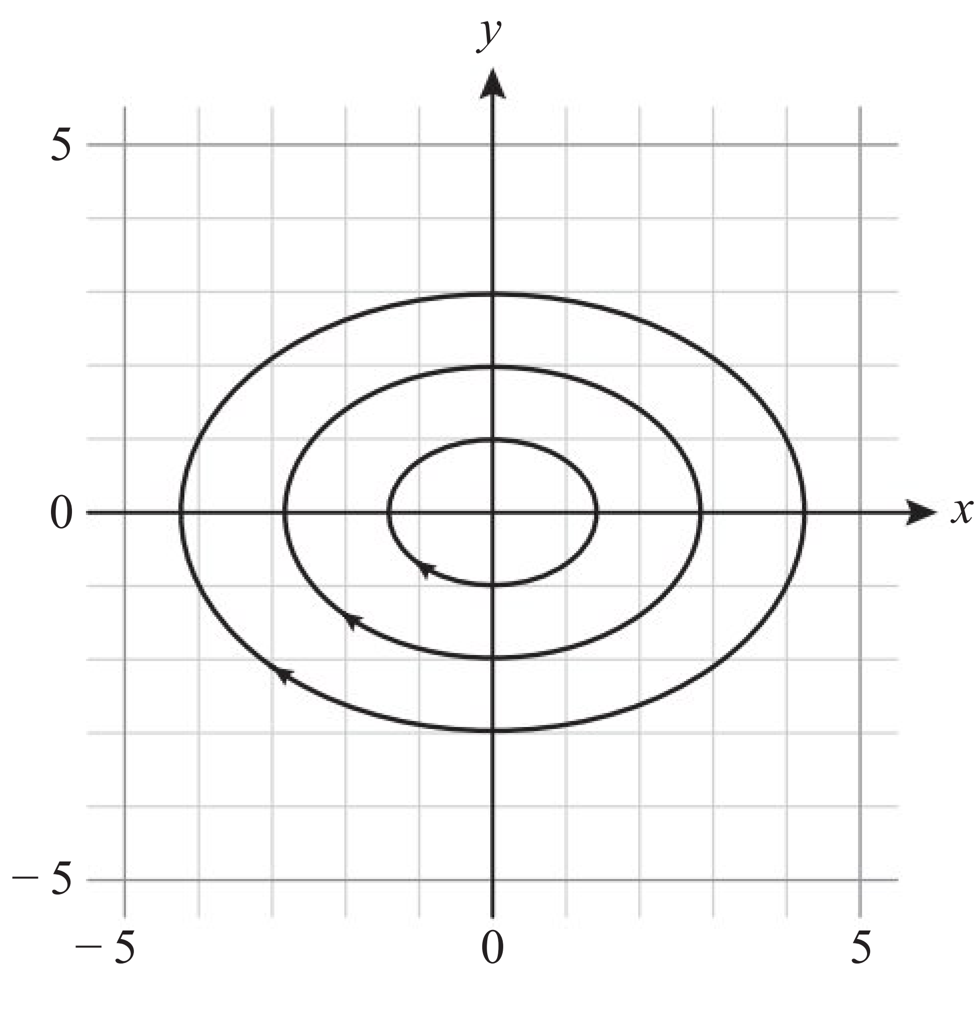
[2 marks]

b $\det\begin{pmatrix} -\lambda & 1 \\ -0.49 & -\lambda \end{pmatrix}$ OR $\lambda^2 + 0.49 = 0$ M1

$\lambda = \pm 0.7i$ A1

Circle in the phase plane A1

Arrow indicating clockwise direction A1



[4 marks]

c The number of fish varies periodically / oscillates M1

Around 400 A1

[2 marks]

Total [8 marks]

18 a H_0: $\mu = 8.7$, H_1 : $\mu > 8.7$ A1

[1 mark]

b $P(X \geq a|\mu = 8.7) < 0.1$ M1

$a = 14$ A1

$P(X < 14|\mu = 9.6)$ M1A1

Note: Award M1 for attempt to find probability with their a and $\mu = 9.6$

$= 0.892$ A1

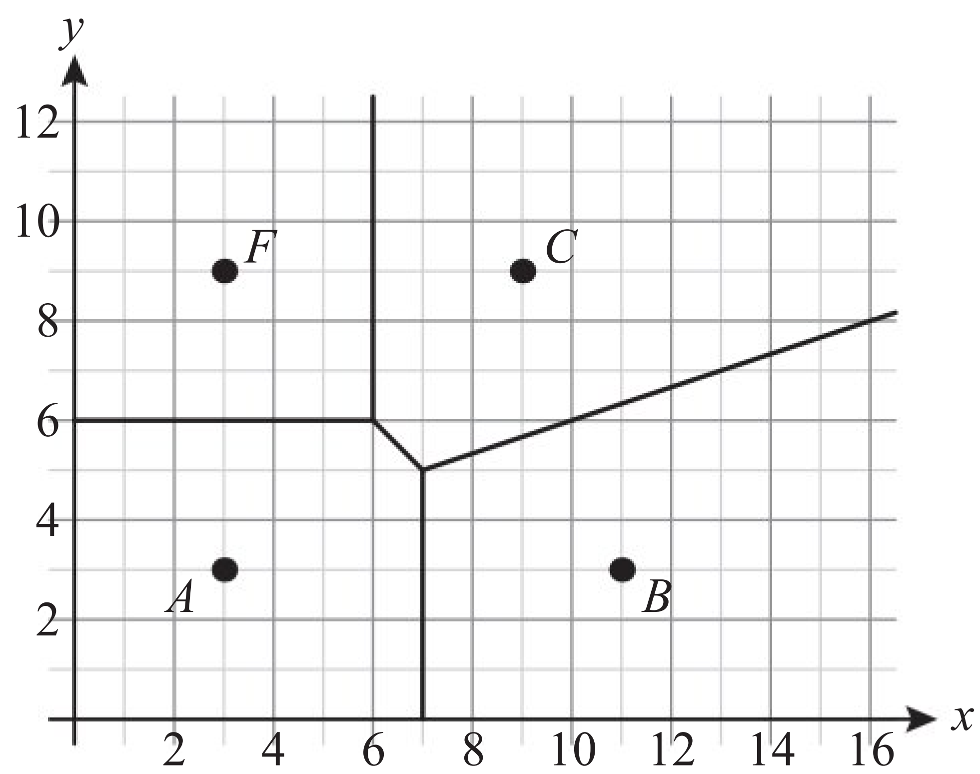
[5 marks]

Total [6 marks]

Practice Set C Paper 2: Mark scheme

1	**a**	Paper 1: mean = 78.9, SD = 17.4	A1	
		Paper 2: mean = 74.0, SD = 15.1	A1	
		Paper 1 has higher marks on average.	A1	
		Paper 2 has more consistent marks.	A1	
				[4 marks]
	b	$r = 0.868$	A1	
		> 0.532	M1	
		There is evidence of positive correlation between the two sets of marks.	A1	
				[3 marks]
	c	**i** Find regression line y on x	M1	
		$y = 0.755x + 14.4$	A1	
		$0.755 \times 95 + 14.4 \approx 86$ marks	A1	
		ii Cannot be used	A1	
		Mark is outside the range of available data (extrapolation)	R1	
				[5 marks]
	d	**i** Boundary for 7: inverse normal of 0.88	M1	
		Boundary = 81	A1	
		5 students	A1	
		ii Use $B(12, 0.12)$	(M1)	
		$P(> 5) = 1 - P(\leq 5)$	(M1)	
		$= 0.00144$	A1	
				[6 marks]
	e	Scaled mark = $\frac{80}{110}$ × original mark	(M1)	
		Mean = 57.4	A1	
		SD = 12.7	A1	
				[3 marks]
				Total [21 marks]
2	**a**	Attempt to differentiate the fraction: obtain $\frac{1}{(x-a)^2}$	M1	
		Obtain $1 - \frac{1}{(x-a)^2}$	A1	
				[2 marks]
	b	Use their $f'(x) > 0$ or set $f'(x) = 0$	(M1)	
		$(3 - a)^2 = 1$	M1	
		$a = 2$ only	A1	
				[3 marks]
	c	$f'(x) = 1 - \frac{1}{2^2} = \frac{3}{4}$	(M1)	
		Gradient of normal is $-\frac{4}{3}$	A1	
		$y - 4.5 = -\frac{4}{3}(x - 4)$	A1	
		Intersect with the curve: $4.5 - \frac{4}{3}(x - 4) = x + \frac{1}{x-2}$	M1	
		Q(2.21, 6.88)	A1	
				[5 marks]
	d	$\int_{2.21}^{4} x + \frac{1}{x-a}\,dx$	M1	
		$= 7.81$	A1	
				[2 marks]
	e	Integrate $\left(x + \frac{1}{x-a}\right)^2$	M1	
		Limits 2.21 and 4	M1	
		Volume $= \pi \int_{2.21}^{4} \left(x + \frac{1}{x-2}\right)^2 dx = 109$	A1	
				[3 marks]
	f	$\int_{4}^{5} x + \frac{1}{x-a}\,dx = \left[\frac{1}{2}x^2 + \ln(x-a)\right]_4^5$	M1	
		$= 4.5 + \ln(5 - a) - \ln(4 - a)$	A1	
		Solve this = 5	(M1)	
		$a = 2.46$	A1	
				[4 marks]
				Total [19 marks]

3	**a**	$\sqrt{2^2 + 6^2}$	(M1)
		$= 6.32\,\text{km}$	A1
			[2 marks]
	b	$\tan^{-1}\left(\frac{2}{6}\right)$ OR $\tan^{-1}\left(\frac{6}{2}\right)$	M1
		$360 - 18.4$ OR $270 + 71.6$	(M1)
		$= 342°$	A1
			[3 marks]
	c	Bisector of AB: $x = 7$	A1
		Midpoint of BC: (10, 6)	A1
		Gradient of BC $= -3$	(M1)
		Bisector of BC: $y - 6 = \frac{1}{3}(x - 10)$	A1
		$x = 7$, solve for y	M1
		$y = 6 + \frac{1}{3}(7 - 10)$	A1
		$x = 7, y = 5$	AG
			[6 marks]
	d	B: $\frac{5 + 8}{2} \times 9$	M1
		$= 58.5\,\text{km}^2$	A1
		C: $(16 \times 12) - A - B$	M1
		$= 74\,\text{km}^2$	A1
			[4 marks]
	e	Lines $x = 6$ and $y = 6$ added	M1
		Correct parts made solid	A1
		Rest of the diagram correct	A1



			[3 marks]
	f	Calculate distance from A or C to (6, 6) and (7, 5)	M1
		$\sqrt{18} < \sqrt{20}$	A1
		The location of the restaurant is at (7, 5)	A1
			[3 marks]
			Total [21 marks]
4	**a**	$\begin{pmatrix} 5 & a \\ -1 & 7 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2a - 5 \\ 15 \end{pmatrix}$	M1
		$(2a - 5, 15)$	A1
			[2 marks]
	b	$\begin{pmatrix} 5 & a \\ -1 & 7 \end{pmatrix}^{-1} = \frac{1}{a + 35}\begin{pmatrix} 7 & -a \\ 1 & 5 \end{pmatrix}$	M1A1
			[2 marks]
	c	$\frac{1}{a + 35}\begin{pmatrix} 7 & -a \\ 1 & 5 \end{pmatrix}\begin{pmatrix} a - 20 \\ 11 \end{pmatrix}$	M1
		$= \frac{1}{a + 35}\begin{pmatrix} -140 - 4a \\ a + 35 \end{pmatrix}$	A1
		P(−4, 1)	A1
			[3 marks]
	d	$\begin{vmatrix} 5 - \lambda & -3 \\ -1 & 7 - \lambda \end{vmatrix} = 0$	M1
		$\lambda = 4, 8$	A1
		Eigenvectors $\begin{pmatrix} 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}$	(M1)A1
			[4 marks]

e	$y = \frac{1}{3}x$ and $y = -x$		A1A1
			[2 marks]
f	Area of $S = 3k$		A1
	det $\mathbf{M} = 32$		A1
	$32 \times 3k = 720$		M1
	$k = 7.5$		A1
			[4 marks]
			Total [17 marks]

5 **a** A–B–D — A1

240 (GBP) — A1

[2 marks]

b At least two correct rows — M1

All correct (Give A1 if rows and columns are swapped.) — A1

0	1	0	1	0	0
0	0	0	1	0	0
0	1	0	0	1	0
1	0	0	0	1	0
0	0	1	0	0	1
0	0	0	0	1	0

[2 marks]

c Cube the adjacency matrix — (M1)

3 ways — A1

[2 marks]

d Consider A, A^2 and A^3 — M1

Looking for all three zero entries — (M1)

A or D — A1

[3 marks]

e A1 for each correct entry.

	B	C	D	
A	128	225	240	A1
B	–	180	112	A1
C	–	–	96	A1

[3 marks]

f **i** Show the circuit with length 561 (e.g. ABDCA) — A1

ii Minimum spanning tree for B, C, D: C–B–D (208) — M1

Add edges AB and AC: 208 + 128 + 225 = 561 — A1

iii The upper and lower bounds are equal, so this is the solution to the travelling salesman problem. — R1

[4 marks]

Total [16 marks]

6 **a** **i** As $R \to 0$, $V \to \infty$ — A1

ii As $R \to \infty$, $V \to 0$ — A1

[2 marks]

b $\frac{A}{r^{12}} - \frac{B}{r^6} = 0$

$Br^6 = A$ — (M1)

$r = \left(\frac{A}{B}\right)^{\frac{1}{6}}$ — A1

[2 marks]

c $\frac{dV}{dr} = -12Ar^{-13} + 6Br^{-7}$ — A1A1

$-12Ar_0^{-13} + 6Br_0^{-7} = 0$ — M1

$-2Ar_0^{-6} + B = 0$

$r_0^{-6} = \frac{B}{2A}$

$r_0 = \left(\frac{2A}{B}\right)^{\frac{1}{6}}$ — A1

[4 marks]

d **i** $\frac{d^2V}{dr^2} = 156Ar^{-14} - 42Br^{-8}$ M1A1

ii Substitute in their value of r_0 M1

$\frac{d^2V}{dr^2} = 2r^{-8}(78Ar^{-6} - 21B)$

$= 2\left(\frac{2A}{B}\right)^{-\frac{4}{3}}\left(78A\left(\frac{2A}{B}\right)^{-1} - 21B\right)$

$= 36B\left(\frac{2A}{B}\right)^{-\frac{4}{3}}$ A1

> 0 since $A, B > 0$, so the point is a local minimum. R1

[5 marks]

e Substitute their equilibrium separation into V M1

$V_{\min} = \frac{1}{\left(\frac{2A}{B}\right)}\left(\frac{A}{\left(\frac{2A}{B}\right)} - B\right)$ A1

$= \frac{B}{2A}\left(\frac{B}{2} - B\right)$ A1

$= -\frac{B^2}{4A}$ AG

[3 marks]

Total [16 marks]

Practice Set C Paper 3: Mark scheme

1 a i $\begin{pmatrix} 6500 \\ -4400 \\ 0 \end{pmatrix} + (t-9)\begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$ (M1)A1

ii e.g. This model assumes that the helicopter is a point mass.
Or it assumes the world is flat R1
Note: Accept any reasonable assumption

[3 marks]

b i When $t = 9$, $B = \begin{pmatrix} 5400 \\ -3500 \\ 10 \end{pmatrix}$ A1

ii The velocity vector is $\begin{pmatrix} 600 \\ -500 \\ 0 \end{pmatrix}$ (M1)

The speed is $\sqrt{600^2 + (-500)^2 + 0^2} = 781 \text{ km hr}^{-1}$ A1

iii e.g. comparing x coordinates: M1

$600t = 6500$ so $t = 10\frac{5}{6}$ (10 : 50) A1

This is not consistent with the z-coordinate (10 vs 5.5) so the two objects do not collide R1

Note: $t = 12\frac{1}{3}$ will be found if z-coordinates are compared first. $t = 10.8$ will be found if y-coordinates are compared first. These should also get full credit if part of a coherent argument.

iv The distance between A and B is (M1)(A1)

$d_{AB} = \sqrt{(600t - 6500)^2 + (1000 - 500t + 4400)^2 + (10 - 3(t-9))^2}$

This can be minimized graphically.
Using the GDC the minimal distance is 13.6 km (3 s.f.) therefore there is no need to provide an alert. R1

[9 marks]

c i $C = \begin{pmatrix} -100 \\ -200 \\ 0 \end{pmatrix} + (100t + 5000t^2)\begin{pmatrix} 1 \\ 2 \\ 0.1 \end{pmatrix}$ A1

Which is of the form of a straight line. R1

ii For a movement of 0.1 up, the horizontal movement is $\sqrt{1^2 + 2^2} = \sqrt{5}$ A1

Therefore the angle of elevation is $\tan^{-1}\left(\frac{0.1}{\sqrt{5}}\right) = 2.56$ (M1)A1

iii The velocity is given by

$$\begin{pmatrix} 100 + 10\,000t \\ 200 + 20\,000t \\ 10 + 1000t \end{pmatrix}$$

The acceleration is given by (A1)

$$\begin{pmatrix} 10\,000 \\ 20\,000 \\ 1000 \end{pmatrix}$$

The magnitude of the acceleration is $\sqrt{10\,000^2 + 20\,000^2 + 1000^2}$ M1
$\approx 22\,383 \text{ km}^{-2}$ A1

[8 marks]

d i $\mathbf{q} = \begin{pmatrix} 100 \sin 3\pi t \\ 100 \cos 3\pi t \\ t \end{pmatrix} - (t)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 100 \sin 3\pi t \\ 100 \cos 3\pi t \\ 0 \end{pmatrix}$ A1

This is the horizontal component of the vector d A1

ii $\boldsymbol{v}_d = \begin{pmatrix} 300\pi \cos 3\pi t \\ -300\pi \sin 2\pi t \\ 1 \end{pmatrix}$ A1

$$\boldsymbol{v}_d \cdot \mathbf{q} = \begin{pmatrix} 300\pi \cos 3\pi t \\ -300\pi \sin 2\pi t \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 100 \sin 3\pi t \\ 100 \cos 3\pi t \\ 0 \end{pmatrix}$$

$= 30\,000\pi \sin 3\pi t \cos 3\pi t - 30\,000\pi \sin 3\pi t \cos 3\pi t + 0 = 0$ A1

Therefore the velocity is always perpendicular to the horizontal component of the displacement, so it is a spiral. R1

[5 marks]

Total [25 marks]

2 **a** **i** Using the quotient rule (M1)

$u = \alpha S$

$w = \beta + S$

$\frac{du}{dS} = \alpha$

$\frac{dw}{dS} = 1$

$\frac{dv}{dS} = \frac{\alpha(\beta + S) - \alpha S}{(\beta + S)^2} = \frac{\alpha\beta}{(\beta + S)^2}$ A1

Since both the numerator and the denominator are positive,
this is a positive quantity so v is increasing as S increases. R1

ii Dividing the top and the bottom of the fraction by S: (M1)

$v = \frac{\alpha}{\frac{\beta}{S} + 1}$

Therefore as $S \to \infty$, $v \to \alpha$ A1

Therefore α is the maximum value of v (when there is an excess of the reactant) R1

iii When $S = \beta$ then $v = \frac{\alpha\beta}{\beta + \beta} = \frac{\alpha}{2}$ A1

So β is the concentration of reactant which is required to get to half of the maximum reaction rate. R1

[8 marks]

b $\frac{1}{v} = \frac{\beta + S}{\alpha S} = \frac{\beta}{\alpha}\frac{1}{S} + \frac{1}{\alpha}$ M1

This is a straight line with gradient $\frac{\beta}{\alpha}$ A1

and intercept $\frac{1}{\alpha}$ A1

[3 marks]

c **i** (A1)

Observation	1/*S*	1/*v*
A	1	0.0556
B	0.2	0.0227
C	0.1	0.0161
D	0.05	0.0128
E	0.0333	0.0123

Line of best fit is $\frac{1}{v} = 0.0442\frac{1}{S} + 0.0117$ A1

ii So $\frac{\beta}{\alpha} = 0.0442$ and $\frac{1}{\alpha} = 0.0117$ (M1)

Therefore $\beta \approx 3.79$ A1

$\alpha \approx 85.7$ A1

[5 marks]

d **i** $r = 0.997$ A1

ii $H_0: \rho = 0$ A1

$H_1: \rho \neq 0$ A1

iii $p = 1.56 \times 10^{-4}$ A1

This is less than 0.05 so there is significant evidence that there is a non-zero underlying correlation. R1

[5 marks]

e If $v_A = 19.8$ then $\frac{1}{v} = 0.0388\frac{1}{S} + 0.0122$ A1

Then $\beta = 3.19$ A1

So the percentage error in the quoted value is $\frac{3.79 - 3.19}{3.19}$ M1

= 19% error. A1

So the error is amplified by linearization. R1

[5 marks]

f The predicted values of y are: (M1)

Observation	$\hat{y}$
A	0.055 919
B	0.020 524
C	0.016 099
D	0.013 887
E	0.013 149

Therefore $MS_E = 2.26 \times 10^{-6}$ (A1)

$SS_X = (1 - 0.277)^2 + (0.2 - 0.277)^2 + \ldots = 0.671$

Therefore the confidence interval for the intercept is

$0.008\,997 < c < 0.014\,352$ (A1)

Since $\alpha = \dfrac{1}{c}$

$69.7 < \alpha < 111$ A1

[4 marks]

Total [30 marks]

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ISBN 978-1-398-32188-5